

The background of the slide is a dark blue field filled with various mathematical symbols and equations in a lighter blue, semi-transparent font. These include $\geq$, H , -1 , $\wedge$, $\frac{\partial}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$, c^2 , $\frac{\partial^2 u}{\partial x^2}$, ψ , $E = mc^2$, $e^{i\pi} + 1 = 0$, $F - E + V = 2$, $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$, $\nabla \cdot \mathbf{B} \approx 0$, $i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \hat{H} \Psi(\mathbf{r}, t)$, $E = mc^2$, and Ψ . A prominent geometric structure, a tetrahedron with internal lines, is rendered in a glowing blue color in the upper left quadrant.

Linear Algebra Notes

Lecture 1:

Systems of Linear Equations

Math Seminar

Reference

These notes follow the textbook *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff, School of Mathematics, Georgia Institute of Technology. Definitions, examples and figures in these slides are taken or adapted from it.

<https://textbooks.math.gatech.edu/ila/>

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Definition. An equation in the unknowns $x, y, z, \dots$ is called *linear* if both sides of the equation are a sum of (constant) multiples of $x, y, z, \dots$, plus an optional constant.

A *system* of linear equations is a collection of several linear equations, like

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2. \end{cases} \quad (1.1.1)$$

Definition (Solution sets).

- A ***solution*** of a system of equations is a list of numbers $x, y, z, \dots$ that make all of the equations true simultaneously.
- The ***solution set*** of a system of equations is the collection of all solutions.
- ***Solving*** the system means finding all solutions with formulas involving some number of parameters.

$$\begin{cases} x + 2y = 3 \\ x + 2y = -3. \end{cases}$$

inconsistent!

In this case, the solution set is *empty*. As this is a rather important property of a system of equations, it has its own name.

Definition. A system of equations is called *inconsistent* if it has no solutions. It is called *consistent* otherwise.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2. \end{cases} \quad (1.1.1)$$

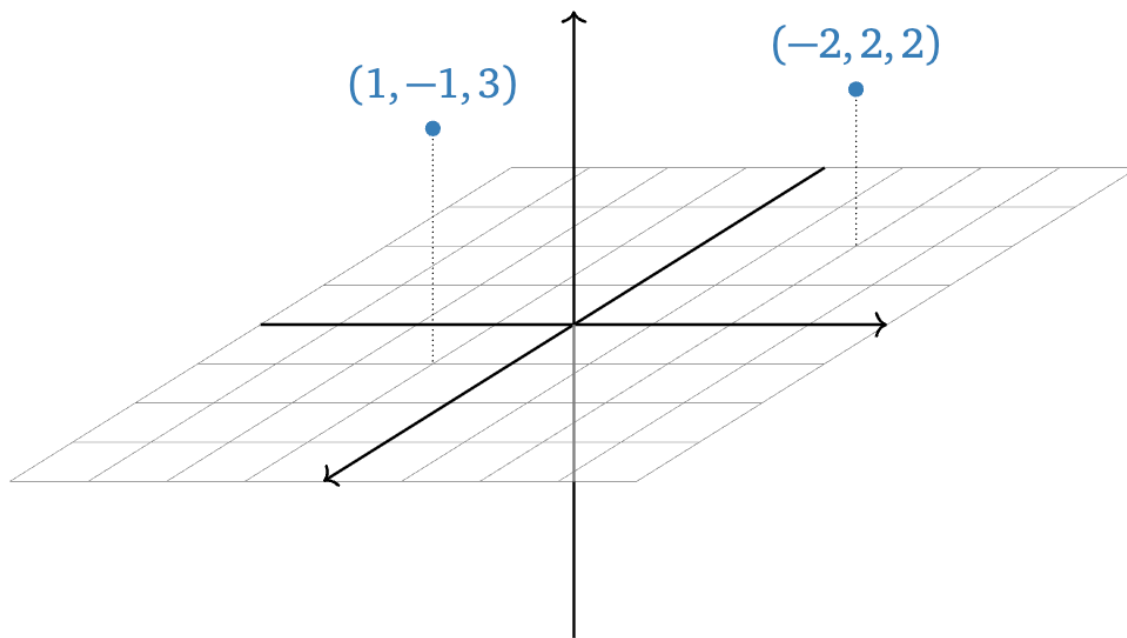
A solution of a system of equations in n variables is a list of n numbers. For example, $(x, y, z) = (1, -2, 3)$ is a solution of (1.1.1). As we will be studying solutions of systems of equations throughout this text, now is a good time to fix our notions regarding lists of numbers.

Definition. Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \dots, x_n).$$

An n -tuple of real numbers is called a ***point*** of $\mathbf{R}^n$.

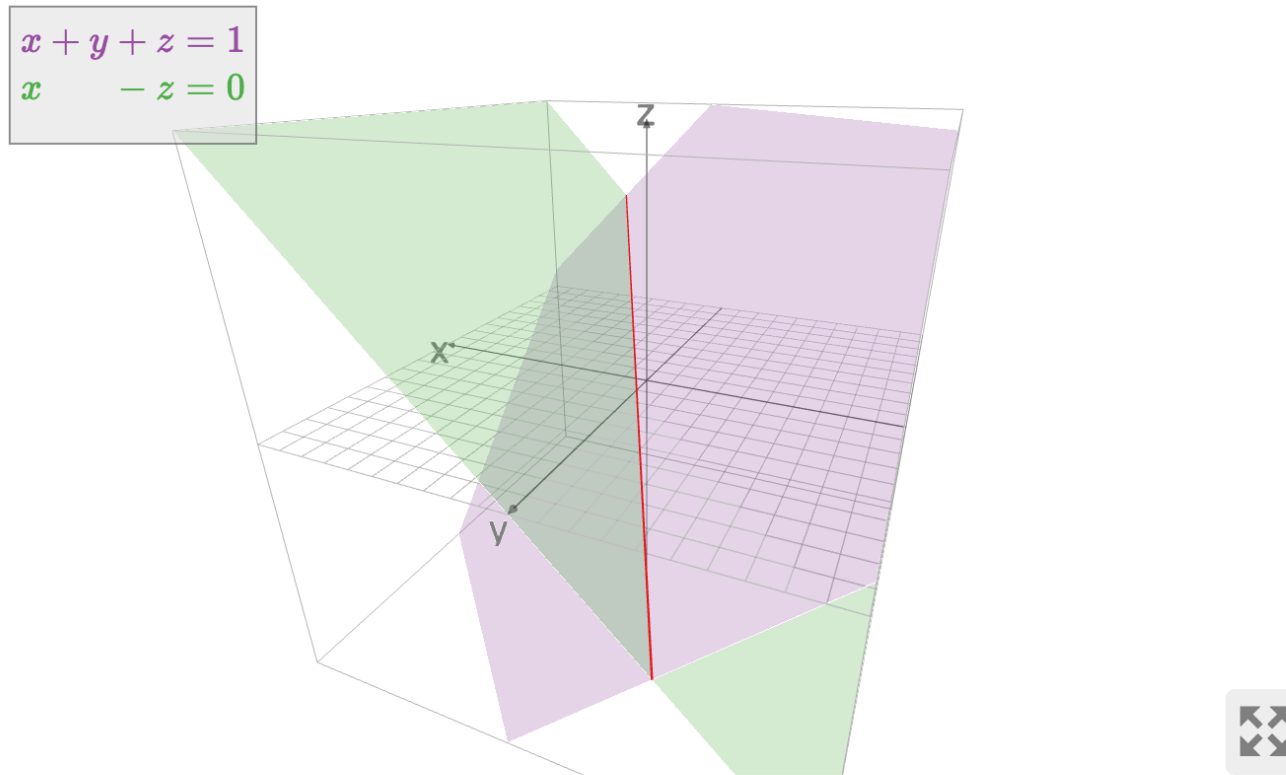
Example (3-Space). When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. We can do so because every point in space can be represented by an ordered triple of real numbers, namely, its x -, y -, and z -coordinates.



Two Equations in Three Variables. Consider the system of two linear equations

$$\begin{cases} x + y + z = 1 \\ x - z = 0. \end{cases}$$

Each equation individually defines a plane in space. The solutions of the system of both equations are the points that lie on both planes. We can see in the picture below that the planes intersect in a line. In particular, this system has infinitely many solutions.



The planes defined by the equations $x + y + z = 1$ and $x - z = 0$ intersect in the red line, which is the solution set of the system of both equations.

Augmented Matrices and Row Operations. Solving equations by elimination requires writing the variables x, y, z and the equals sign $=$ over and over again, merely as placeholders: all that is changing in the equations is the coefficient *numbers*. We can make our life easier by extracting only the numbers, and putting them in a box:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \xrightarrow{\text{becomes}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right).$$

This is called an *augmented matrix*. The word “augmented” refers to the vertical line, which we draw to remind ourselves where the equals sign belongs; a *matrix* is a grid of numbers without the vertical line. In this notation, our three valid ways of manipulating our equations become *row operations*: **see next slide**

- **Scaling:** multiply all entries in a row by a nonzero number.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right) \xrightarrow{R_1 = R_1 \times -3} \left(\begin{array}{ccc|c} -3 & -6 & -9 & -18 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

Here the notation R_1 simply means “the first row”, and likewise for R_2, R_3 , etc.

- **Replacement:** add a multiple of one row to another, replacing the second row with the result.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right) \xrightarrow{R_2 = R_2 - 2 \times R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

- **Swap:** interchange two rows.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 3 & 1 & -1 & -2 \\ 2 & -3 & 2 & 14 \\ 1 & 2 & 3 & 6 \end{array} \right)$$

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition. Two matrices are called *row equivalent* if one can be obtained from the other by doing some number of row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Solve the following system of equations using row operations:

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases}$$

Solution

First we put our system of equations into an augmented matrix.

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array} \right)$$

We clear the entries below the top-left using row replacement.

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 3 & 4 & | & 5 \\ 4 & 5 & | & 9 \end{pmatrix} \xrightarrow{R_2=R_2-3R_1} \begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 4 & 5 & | & 9 \end{pmatrix}$$
$$\xrightarrow{R_3=R_3-4R_1} \begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 1 & | & 1 \end{pmatrix}$$

Now we clear the second entry from the last row.

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 1 & | & 1 \end{pmatrix} \xrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 0 & | & 2 \end{pmatrix}$$

This translates back into the system of equations

$$\begin{cases} x + y = 2 \\ y = -1 \\ 0 = 2. \end{cases}$$

Our original system has the same solution set as this system. But this system has no solutions: there are no values of x, y making the third equation true! We conclude that our original equation was inconsistent.

Definition. A matrix is in *row echelon form* if:

1. All zero rows are at the bottom.
2. The first nonzero entry of a row is to the *right* of the first nonzero entry of the row above.
3. Below the first nonzero entry of a row, all entries are zero.

Here is a picture of a matrix in row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star = \text{any number}$
 $\boxed{\star} = \text{any nonzero number}$

Definition. A *pivot* is the first nonzero entry of a row of a matrix in row echelon form.

A matrix in row-echelon form is generally easy to solve using back-substitution. For example,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right) \xrightarrow{\text{becomes}} \begin{cases} x + 2y + 3z = 6 \\ y + 2z = 4 \\ 10z = 30. \end{cases}$$

We immediately see that $z = 3$, which implies $y = 4 - 2 \cdot 3 = -2$ and $x = 6 - 2(-2) - 3 \cdot 3 = 1$. See this [example](#).

Definition. A matrix is in *reduced row echelon form* if it is in row echelon form, and in addition:

4. Each pivot is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Here is a picture of a matrix in reduced row echelon form:

$$\begin{pmatrix} \color{red}{1} & 0 & \star & 0 & \star \\ 0 & \color{red}{1} & \star & 0 & \star \\ 0 & 0 & 0 & \color{red}{1} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \star = \text{any number} \\ \color{red}{1} = \text{pivot} \end{array}$$

A matrix in reduced row echelon form is in some sense completely solved. For example,

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right) \xrightarrow{\text{becomes}} \begin{cases} x = 1 \\ y = -2 \\ z = 3. \end{cases}$$

Theorem. Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

We will give an algorithm, called **row reduction** or **Gaussian elimination**, which demonstrates that every matrix is row equivalent to *at least one* matrix in reduced row echelon form.

The uniqueness statement is interesting—it means that, no matter *how* you row reduce, you *always* get the same matrix in reduced row echelon form.

This assumes, of course, that you only do the three legal row operations, and you don't make any arithmetic errors.

The Row Echelon Form of an Inconsistent System. *An augmented matrix corresponds to an inconsistent system of equations if and only if the last column (i.e., the augmented column) is a pivot column.*

In other words, the row reduced matrix of an inconsistent system looks like this:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Example (A System with a Free Variable). Consider the linear system

$$\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1. \end{cases}$$

We solve it using row reduction:

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} \color{red}{1} & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array} \right) \quad \text{(Optional)}$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ \color{red}{0} & -3 & -6 & 3 \end{array} \right) \quad \text{(Step 1c)}$$

$$\xrightarrow{R_2 = R_2 \div -3} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & \color{red}{1} & 2 & -1 \end{array} \right) \quad \text{(Step 2b)}$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & \color{red}{0} & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right) \quad \text{(Step 2c)}$$

This row reduced matrix corresponds to the linear system

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1. \end{cases}$$

In what sense is the system solved? We rewrite as

$$\begin{cases} x = 1 - 5z \\ y = -1 - 2z. \end{cases}$$

For any value of z , there is exactly one value of x and y that make the equations true. But we are free to choose *any* value of z .

We have found all solutions: it is the set of all values x, y, z , where

$$\begin{cases} x = 1 - 5z \\ y = -1 - 2z \\ z = z \end{cases} \quad z \text{ any real number.}$$

This is called the *parametric form* for the solution to the linear system. The variable z is called a *free variable*.

Given the parametric form for the solution to a linear system, we can obtain specific solutions by replacing the free variables with any specific real numbers. For instance, setting $z = 0$ in the last example gives the solution $(x, y, z) = (1, -1, 0)$, and setting $z = 1$ gives the solution $(x, y, z) = (-4, -3, 1)$.

Definition. Consider a *consistent* system of equations in the variables $x_1, x_2, \dots, x_n$. Let A be a row echelon form of the augmented matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

In the above example, the variable z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right).$$

In the matrix

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array} \right),$$

the free variables are x_2 and x_4 . (The augmented column is not free because it does not correspond to a variable.)

Implicit Versus Parameterized Equations. The solution set of the system of linear equations

$$\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$$

is a line in $\mathbf{R}^3$, as we saw in this [example](#). These equations are called the ***implicit equations*** for the line: the line is defined *implicitly* as the simultaneous solutions to those two equations.

The parametric form

$$\begin{cases} x = 1 - 5z \\ y = -1 - 2z. \end{cases}$$

can be written as follows:

$$(x, y, z) = (1 - 5z, -1 - 2z, z) \quad z \text{ any real number.}$$

This called a ***parameterized equation*** for the same line. It is an expression that produces all points of the line in terms of one parameter, z .

One should think of a system of equations as being an implicit equation for its solution set, and of the parametric form as being the parameterized equation for the same set. The parameteric form is much more explicit: it gives a concrete recipe for producing *all* solutions.

You can choose *any value* for the free variables in a (consistent) linear system.

Free variables come from the *columns without pivots* in a matrix in row echelon form.

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. ***The last column is a pivot column.*** In this case, the system is *inconsistent*. There are zero solutions, i.e., the solution set is empty. For example, the matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

comes from a linear system with no solutions.

2. ***Every column except the last column is a pivot column.*** In this case, the system has a *unique* solution. For example, the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right)$$

tells us that the unique solution is $(x, y, z) = (a, b, c)$.

3. ***The last column is not a pivot column, and some other column is not a pivot column either.*** In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). For example, in the system corresponding to the matrix

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & -1 \end{array} \right),$$

any values for x_2 and x_4 yield a solution to the system of equations.