

The background of the slide is a dark blue field filled with various mathematical symbols and equations in a lighter blue, semi-transparent font. These include symbols like $\geq$, H , -1 , $\wedge$, $\&$, $e^{i\pi} + 1 = 0$, $F - E + V = 2$, $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$, $\nabla \cdot \mathbf{B} \approx 0$, $E = mc^2$, ψ , Δ , ∇ , γ , τ , χ , λ , μ , ν , ξ , η , θ , ϕ , σ , ω , ρ , κ , ι , $\jmath$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{o}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$, $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$, $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$. A prominent geometric diagram in the center features a network of lines connecting several points, with a yellow line segment highlighted. The overall aesthetic is technical and academic.

Linear Algebra Notes

Lecture 2:

Geometry of Solutions

Math Seminar

Reference

These notes follow the textbook *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff, School of Mathematics, Georgia Institute of Technology. Definitions, examples and figures in these slides are taken or adapted from it.

<https://textbooks.math.gatech.edu/ila/>

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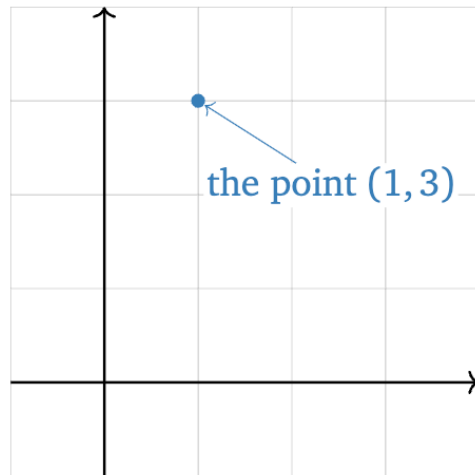
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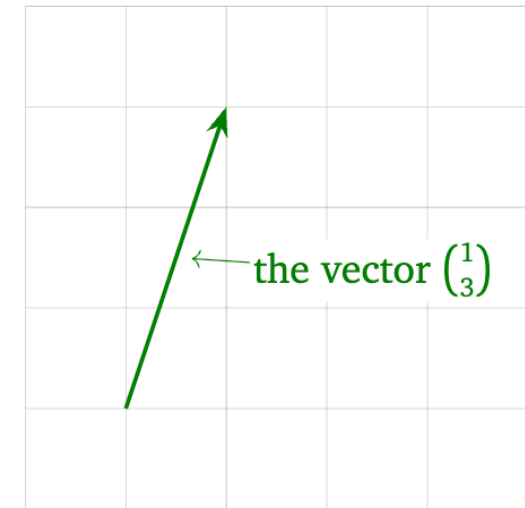
Vectors in $\mathbf{R}^n$

We have been drawing points in $\mathbf{R}^n$ as dots in the line, plane, space, etc. We can also draw them as *arrows*. Since we have two geometric interpretations in mind, we now discuss the relationship between the two points of view.

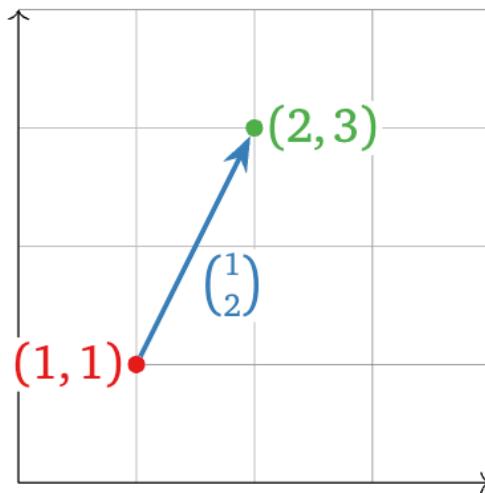
Points and Vectors. Again, a *point* in $\mathbf{R}^n$ is drawn as a dot.



A *vector* is a point in $\mathbf{R}^n$, drawn as an arrow.



Note. Another way to think about a vector is as a *difference* between two points, or the arrow from one point to another. For instance, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the arrow from $(1, 1)$ to $(2, 3)$.



Vector addition and scalar multiplication.

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + x \\ b + y \\ c + z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number c :

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

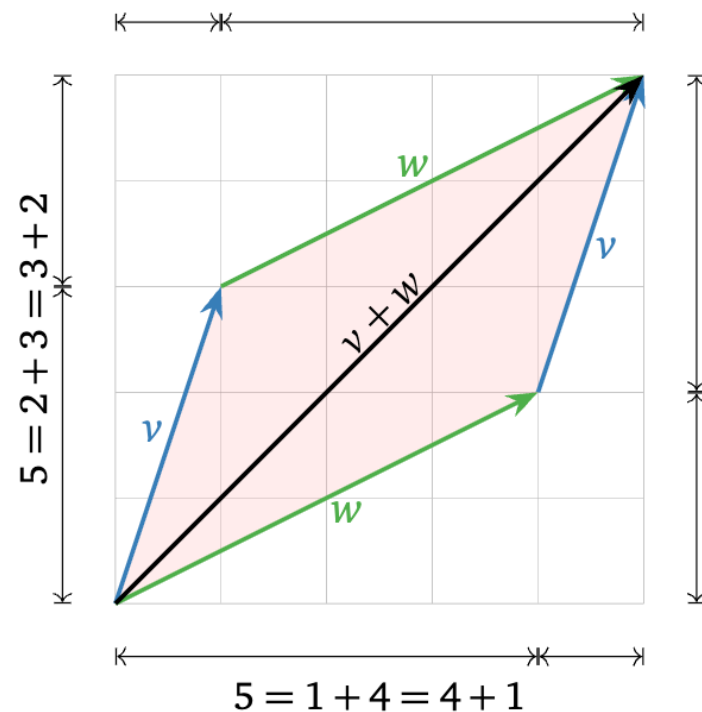
We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, then cv is called a **scalar multiple** of v .

Addition and scalar multiplication work in the same way for vectors of length n .

The Parallelogram Law for Vector Addition. Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v . Then $v + w$ is the vector whose tail is the tail of v and whose head is the head of w . Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

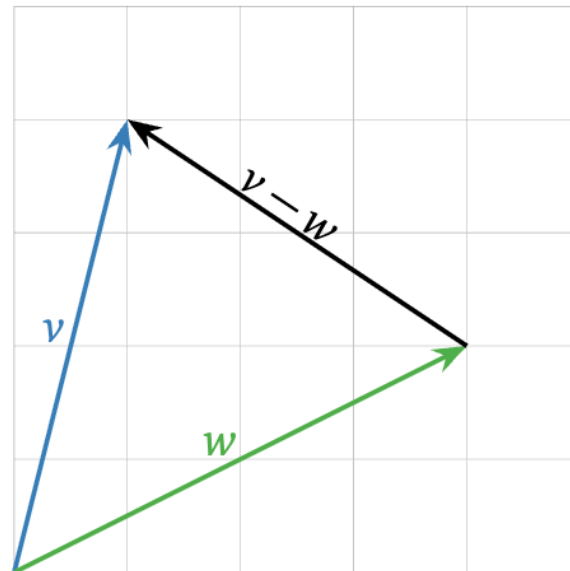
Why? The width of $v + w$ is the sum of the widths, and likewise with the heights.



Vector Subtraction. Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of w to the head of v . For example,

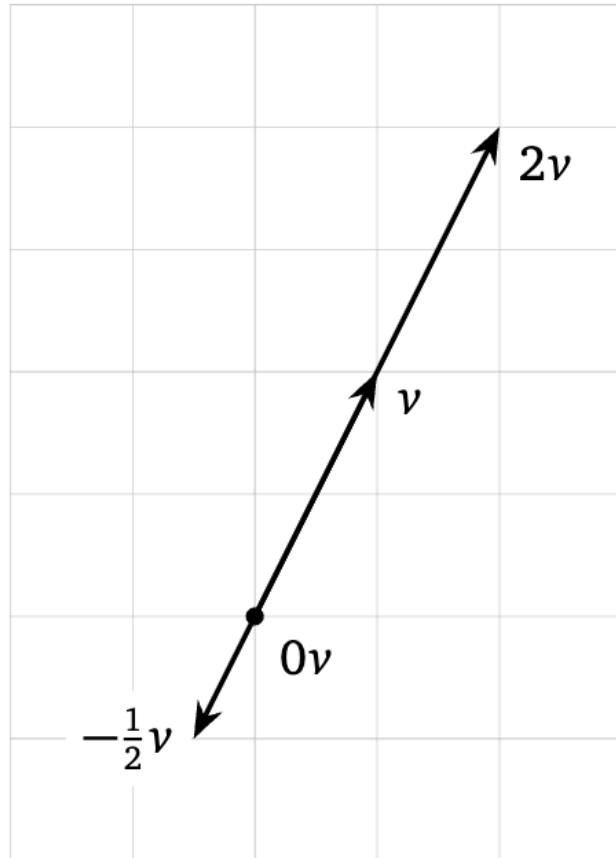
$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Why? If you add $v - w$ to w , you get v .

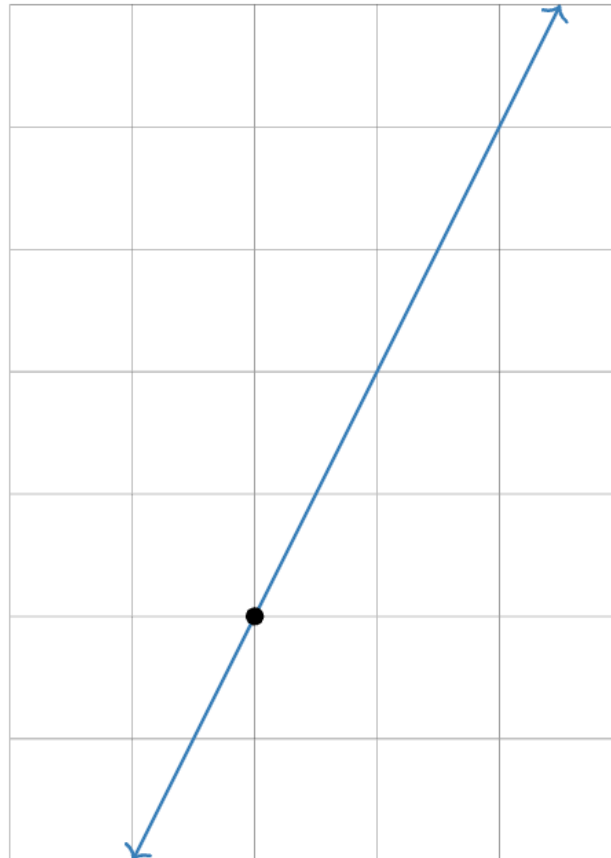


Scalar Multiplication. A scalar multiple of a vector v has the same (or opposite) direction, but a different length. For instance, $2v$ is the vector in the direction of v but twice as long, and $-\frac{1}{2}v$ is the vector in the opposite direction of v , but half as long. Note that the set of all scalar multiples of a (nonzero) vector v is a *line*.

Some multiples of v .



All multiples of v .



Linear Combinations

We can add and scale vectors in the same equation.

Definition. Let $c_1, c_2, \dots, c_k$ be scalars, and let $v_1, v_2, \dots, v_k$ be vectors in $\mathbf{R}^n$. The vector in $\mathbf{R}^n$

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

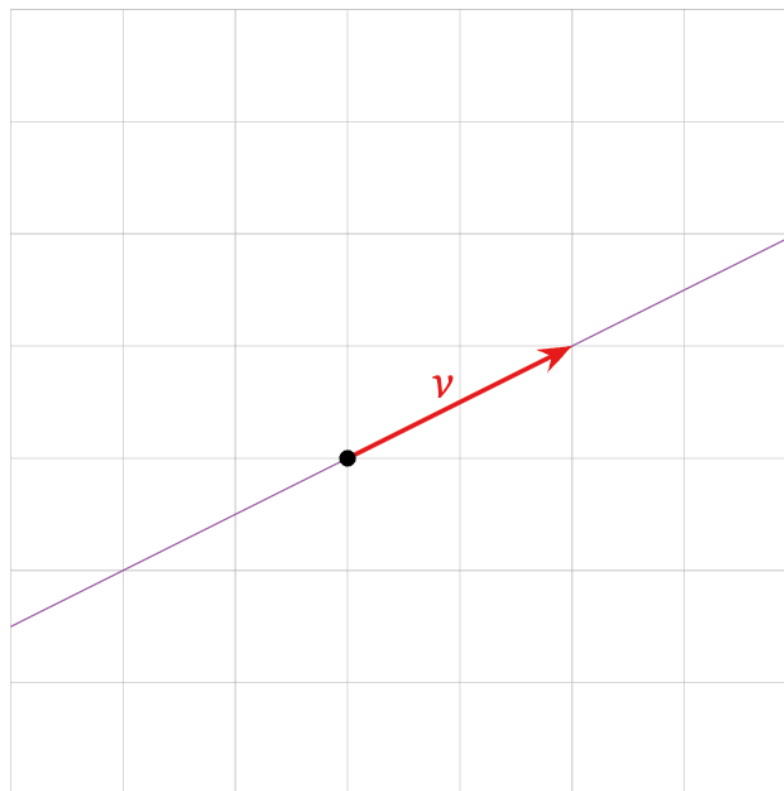
is called a ***linear combination*** of the vectors $v_1, v_2, \dots, v_k$, with ***weights*** or ***coefficients*** $c_1, c_2, \dots, c_k$.

Geometrically, a linear combination is obtained by stretching / shrinking the vectors $v_1, v_2, \dots, v_k$ according to the coefficients, then adding them together using the parallelogram law.

Example (Linear Combinations of a Single Vector). A linear combination of a single vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is just a scalar multiple of v . So some examples include

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \frac{3}{2}v = \begin{pmatrix} 3/2 \\ 3 \end{pmatrix}, \quad -\frac{1}{2}v = \begin{pmatrix} -1/2 \\ -1 \end{pmatrix}, \quad \dots$$

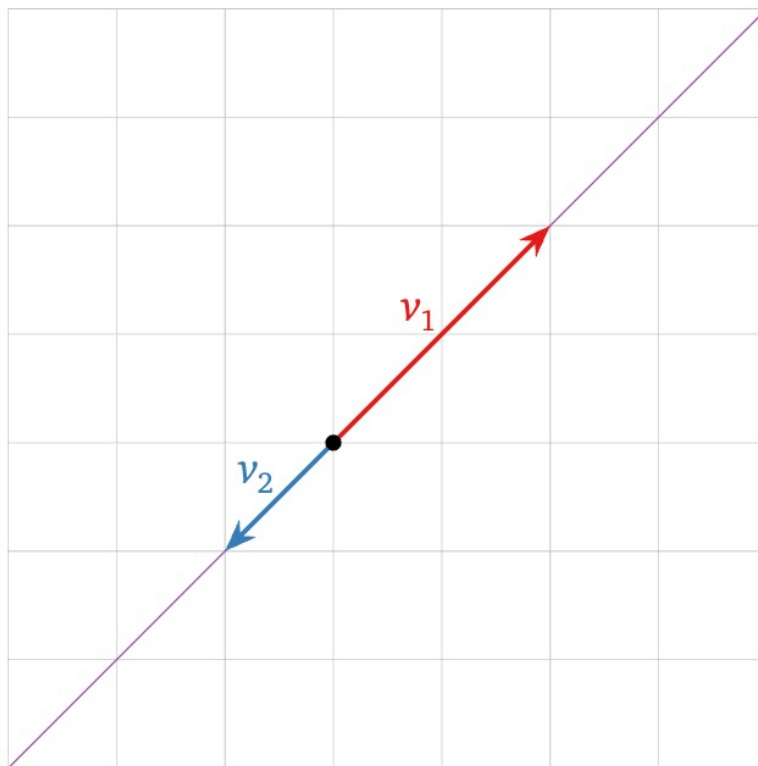
The set of all linear combinations is the *line through* v . (Unless $v = 0$, in which case any scalar multiple of v is again 0.)



Example (Linear Combinations of Collinear Vectors). The set of all linear combinations of the vectors

$$v_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

is the *line* containing both vectors.



The difference between this and a previous example is that both vectors lie on the same line. Hence any scalar multiples of v_1, v_2 lie on that line, as does their sum.

An equation involving vectors with n coordinates is the same as n equations involving only numbers. For example, the equation

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

simplifies to

$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

For two vectors to be equal, all of their coordinates must be equal, so this is just the system of linear equations

$$\begin{cases} x - y = 8 \\ 2x - 2y = 16 \\ 6x - y = 3. \end{cases}$$

Definition. A *vector equation* is an equation involving a linear combination of vectors with possibly unknown coefficients.

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Asking whether or not a vector equation has a solution is the same as asking if a given vector is a linear combination of some other given vectors.

For example the vector equation above is asking if the vector $(8, 16, 3)$ is a linear combination of the vectors $(1, 2, 6)$ and $(-1, 2, -1)$.

Now we have *three* equivalent ways of thinking about a linear system:

1. As a system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array} \right)$$

3. As a vector equation ($x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

The third is geometric in nature: it lends itself to drawing pictures.

Spans

It will be important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \dots, v_k$ in $\mathbf{R}^n$. In other words, we would like to understand the set of all vectors b in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \dots, x_k$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = b$$

has a solution (i.e. is consistent).

 **Definition.** Let $v_1, v_2, \dots, v_k$ be vectors in $\mathbf{R}^n$. The ***span*** of $v_1, v_2, \dots, v_k$ is the collection of all linear combinations of $v_1, v_2, \dots, v_k$, and is denoted $\text{Span}\{v_1, v_2, \dots, v_k\}$. In symbols:

$$\text{Span}\{v_1, v_2, \dots, v_k\} = \{x_1 v_1 + x_2 v_2 + \cdots + x_k v_k \mid x_1, x_2, \dots, x_k \text{ in } \mathbf{R}\}$$

We also say that $\text{Span}\{v_1, v_2, \dots, v_k\}$ is the subset ***spanned by*** or ***generated by*** the vectors $v_1, v_2, \dots, v_k$.

Three characterizations of consistency. Now we have three equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \dots, v_k$.

2. The vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_k v_k = b$$

has a solution.

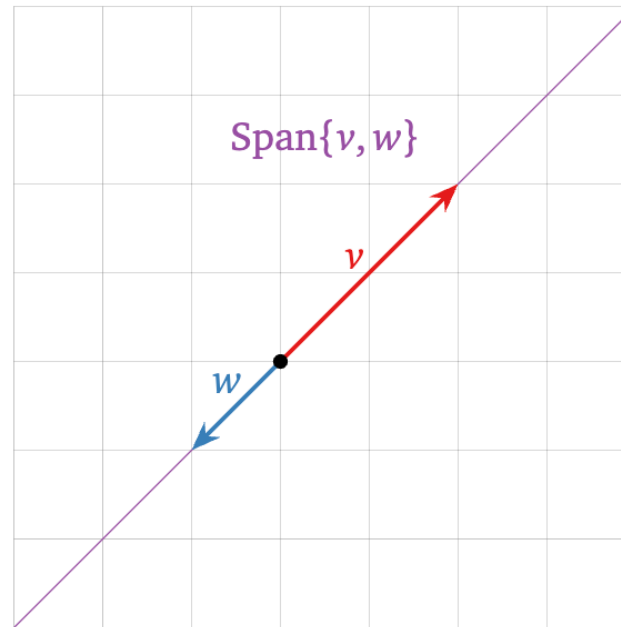
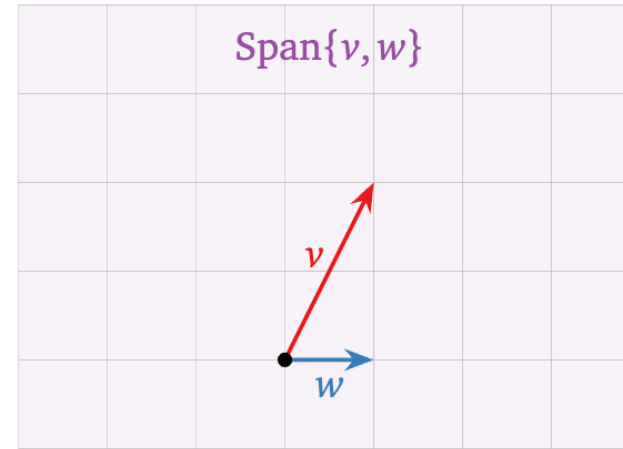
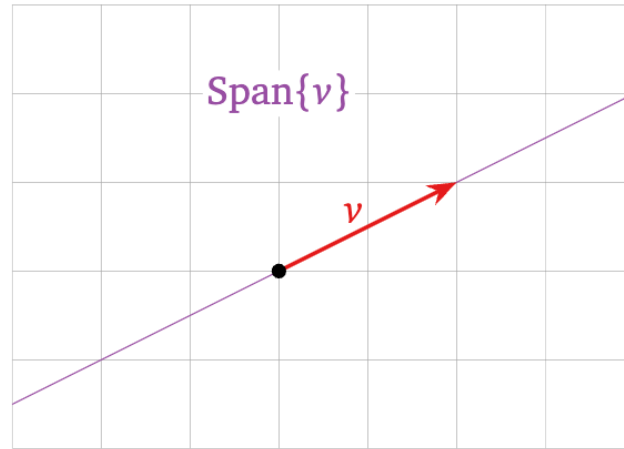
3. The linear system with augmented matrix

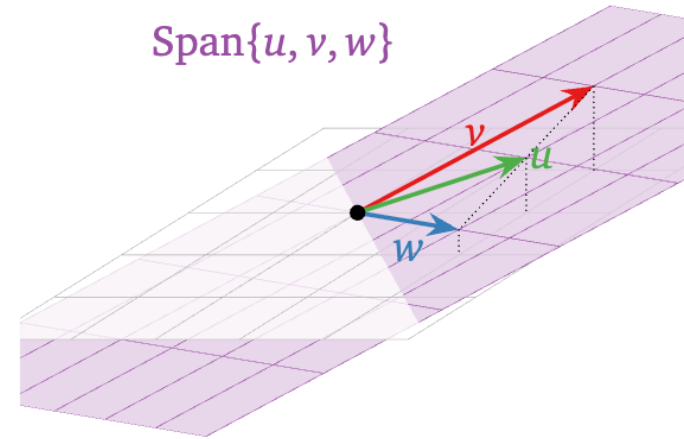
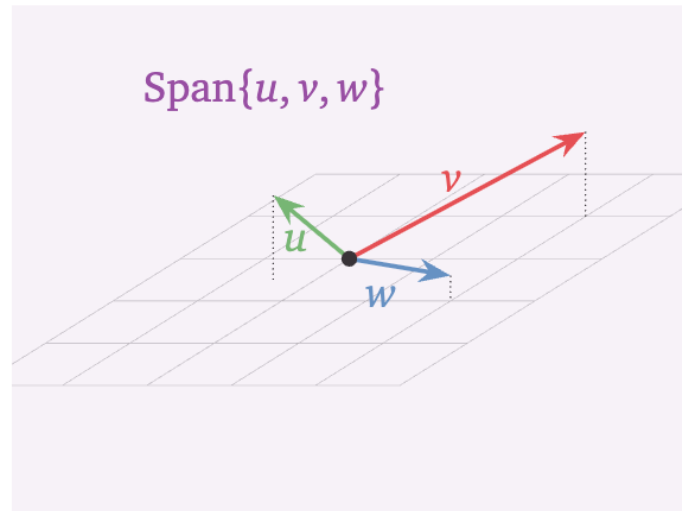
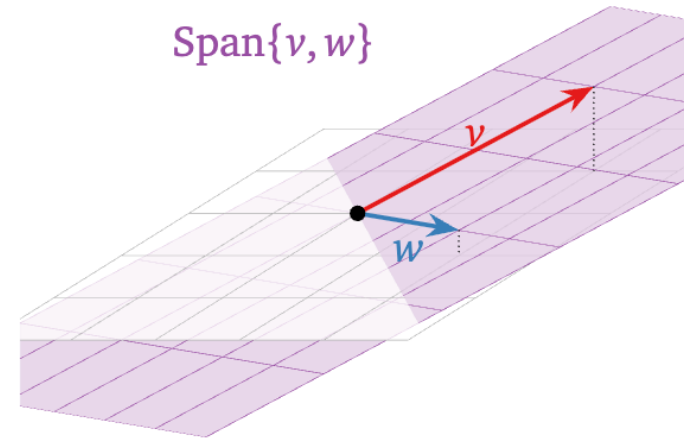
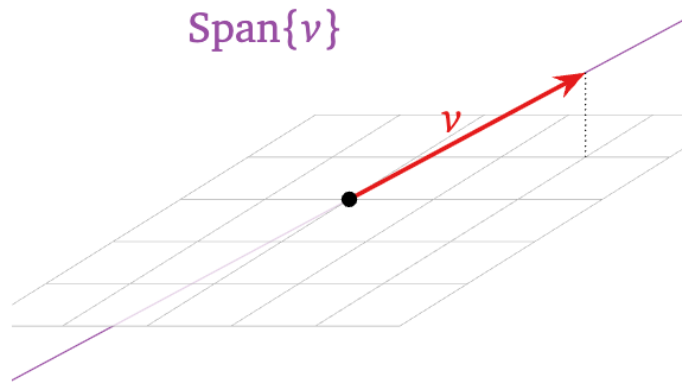
$$\left(\begin{array}{c|c|c|c|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array} \right)$$

is consistent.

Equivalent means that, for any given list of vectors $v_1, v_2, \dots, v_k$, b , either all three statements are true, or all three statements are false.

Pictures of Spans. Drawing a picture of $\text{Span}\{v_1, v_2, \dots, v_k\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \dots, v_k$.





Pictures of spans in $\mathbf{R}^3$. The span of two noncollinear vectors is the plane containing the origin and the heads of the vectors. Note that three coplanar (but not collinear) vectors span a plane and not a 3-space, just as two collinear vectors span a line and not a plane.

The Matrix Equation $Ax = b$.

In this section we introduce a very concise way of writing a system of linear equations: $Ax = b$. Here A is a matrix and x, b are vectors (generally of different sizes), so first we must explain how to multiply a matrix by a vector.

When we say “ A is an $m \times n$ matrix,” we mean that A has m rows and n columns.

Definition. Let A be an $m \times n$ matrix with columns $v_1, v_2, \dots, v_n$:

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is a vector in $\mathbf{R}^m$.

If A is an $m \times n$ matrix (m rows, n columns), then Ax makes sense when x has n entries. The product Ax has m entries.

Properties of the Matrix-Vector Product. *Let A be an $m \times n$ matrix, let u, v be vectors in $\mathbf{R}^n$, and let c be a scalar. Then:*

- $A(u + v) = Au + Av$
- $A(cu) = cAu$

Definition. A *matrix equation* is an equation of the form $Ax = b$, where A is an $m \times n$ matrix, b is a vector in $\mathbf{R}^m$, and x is a vector whose coefficients $x_1, x_2, \dots, x_n$ are unknown.

In this book we will study two complementary questions about a matrix equation $Ax = b$:

1. Given a specific choice of b , what are all of the solutions to $Ax = b$?
2. What are all of the choices of b so that $Ax = b$ is consistent?

The first question is more like the questions you might be used to from your earlier courses in algebra; you have a lot of practice solving equations like $x^2 - 1 = 0$ for x . The second question is perhaps a new concept for you. The rank theorem in Section 2.9, which is the culmination of this chapter, tells us that the two questions are intimately related.

Matrix Equations and Vector Equations. Let $v_1, v_2, \dots, v_n$ and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

This is equivalent to the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then $Ax = b$ is equivalent to the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b,$$

where $v_1, v_2, \dots, v_n$ are the columns of A , and $x_1, x_2, \dots, x_n$ are the entries of x .

Four Ways of Writing a Linear System. We now have *four* equivalent ways of writing (and thinking about) a system of linear equations:

1. As a system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array} \right)$$

3. As a vector equation ($x_1v_1 + x_2v_2 + \cdots + x_nv_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

In particular, *all four have the same solution set.*

Spans and Consistency

Let A be a matrix with columns $v_1, v_2, \dots, v_n$:

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}.$$

Then

$Ax = b$ has a solution

$$\iff \text{there exist } x_1, x_2, \dots, x_n \text{ such that } A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

$$\iff \text{there exist } x_1, x_2, \dots, x_n \text{ such that } x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

$$\iff b \text{ is a linear combination of } v_1, v_2, \dots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

Spans and Consistency.

The matrix equation $Ax = b$ has a solution if and only if b is in the span of the columns of A .

This gives an equivalence between an *algebraic* statement ($Ax = b$ is consistent), and a *geometric* statement (b is in the span of the columns of A).

When Solutions Always Exist. Building on this note, we have the following criterion for when $Ax = b$ is consistent for *every* choice of b .

Theorem. Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution for all b in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot position in every row.

Homogeneous Systems [¶ permalink](#)

The equation $Ax = b$ is easier to solve when $b = 0$, so we start with this case.

Definition. A system of linear equations of the form $Ax = 0$ is called *homogeneous*.

A system of linear equations of the form $Ax = b$ for $b \neq 0$ is called *inhomogeneous*.

A homogeneous system is just a system of linear equations where all constants on the right side of the equals sign are zero.

A homogeneous system always has the solution $x = 0$. This is called the *trivial solution*. Any nonzero solution is called *nontrivial*.

Observation. The equation $Ax = 0$ has a nontrivial solution $\iff$ there is a free variable $\iff$ A has a column without a pivot position.

Parametric Vector Form (homogeneous case). Consider the following matrix in reduced row echelon form:

$$A = \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix equation $Ax = 0$ corresponds to the system of equations

$$\begin{cases} x_1 - 8x_3 - 7x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0. \end{cases}$$

We can write the parametric form as follows:

$$\begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4. \end{cases}$$

We wrote the redundant equations $x_3 = x_3$ and $x_4 = x_4$ in order to turn the above system into a *vector equation*:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

This vector equation is called the ***parametric vector form*** of the solution set. Since x_3 and x_4 are allowed to be anything, this says that the solution set is the set of all

linear combinations of $\begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$. In other words, the solution set is

$$\text{Span} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The set of solutions to a homogeneous equation $Ax = 0$ is a span.

Inhomogeneous Systems

Recall that a matrix equation $Ax = b$ is called *inhomogeneous* when $b \neq 0$.

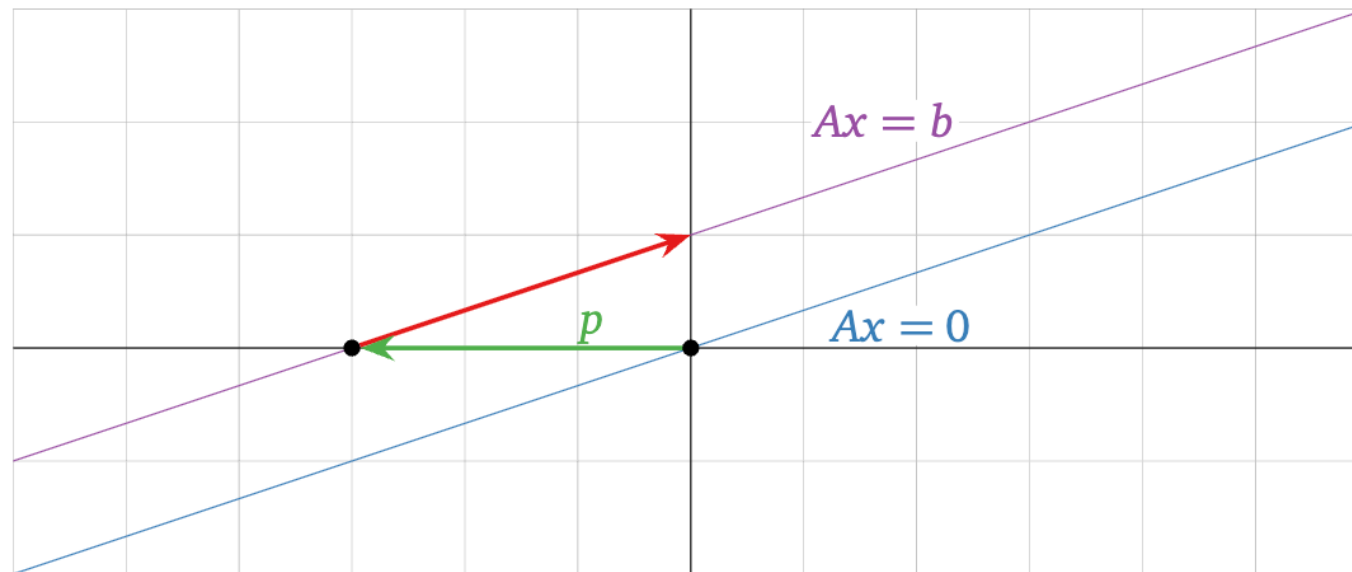
Example (The solution set is a line). ▼

In the above example, the solution set was all vectors of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

where x_2 is any scalar. The vector $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ is also a solution of $Ax = b$: take $x_2 = 0$. We call p a *particular solution*.

In the solution set, x_2 is allowed to be anything, and so the solution set is obtained as follows: we take all scalar multiples of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and then add the particular solution $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ to each of these scalar multiples. Geometrically, this is accomplished by first drawing the span of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, which is a line through the origin (and, not coincidentally, the solution to $Ax = 0$), and we *translate*, or push, this line along $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$. The translated line contains p and is parallel to $\text{Span}\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\}$: it is a *translate of a line*.



Key Observation.

If $Ax = b$ is consistent, the set of solutions to is obtained by taking one *particular solution* p of $Ax = b$, and adding all solutions of $Ax = 0$.

In particular, if $Ax = b$ is consistent, the solution set is a *translate of a span*.

To every $m \times n$ matrix A , we have now associated two completely different geometric objects, both described using spans.

- The ***solution set***: for fixed b , this is the set of all x such that $Ax = b$.
 - This is a span if $b = 0$, and it is a translate of a span if $b \neq 0$ (and $Ax = b$ is consistent).
 - It is a subset of $\mathbf{R}^n$.
 - It is computed by solving a system of equations: usually by row reducing and finding the parametric vector form.
- The ***span of the columns of A*** : this is the set of all b such that $Ax = b$ is consistent.
 - This is always a span.
 - It is a subset of $\mathbf{R}^m$.
 - It is not computed by solving a system of equations: row reduction plays no role.