

Linear Algebra Notes

Lecture 3:

Linear Independence

Reference

These notes follow the textbook *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff, School of Mathematics, Georgia Institute of Technology. Definitions, examples and figures in these slides are taken or adapted from it.

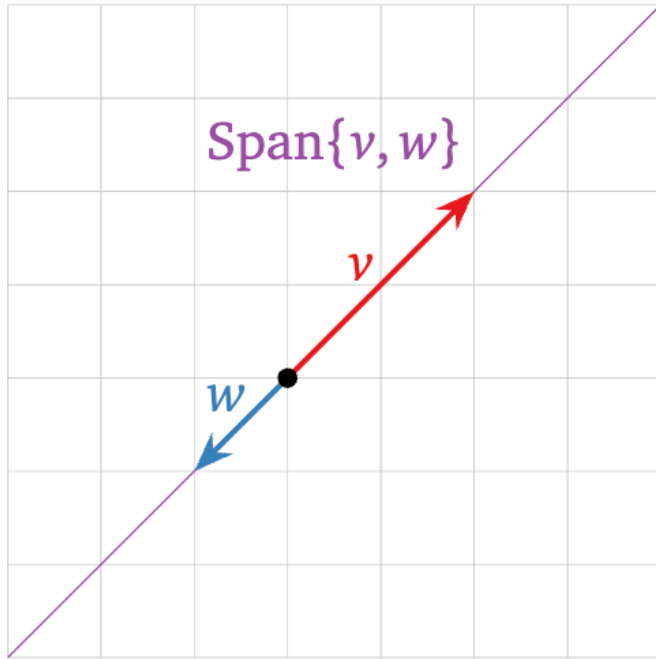
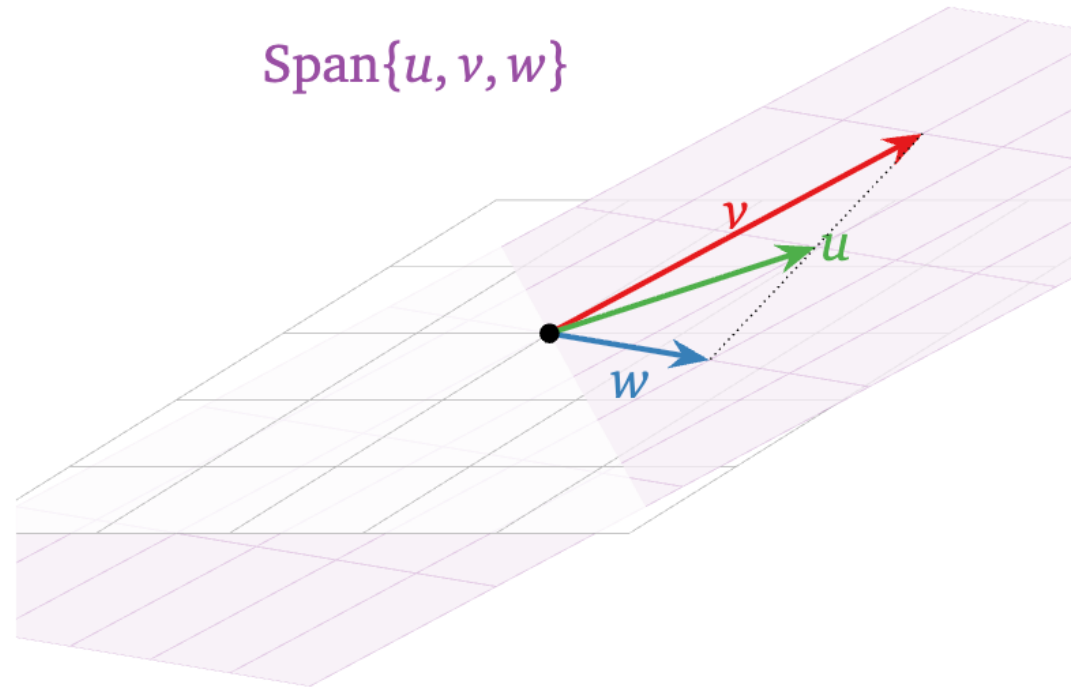
<https://textbooks.math.gatech.edu/ila/>

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Sometimes the span of a set of vectors is “smaller” than you expect from the number of vectors, as in the picture below. This means that (at least) one of the vectors is redundant: it can be removed without affecting the span. In the present section, we formalize this idea in the notion of *linear independence*.



Pictures of sets of vectors that are linearly dependent. Note that in each case, one vector is in the span of the others—so it doesn't make the span bigger.

 **Definition.** A set of vectors $\{v_1, v_2, \dots, v_k\}$ is *linearly independent* if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_k v_k = 0$$

has only the trivial solution $x_1 = x_2 = \dots = x_k = 0$. The set $\{v_1, v_2, \dots, v_k\}$ is *linearly dependent* otherwise.

In other words, $\{v_1, v_2, \dots, v_k\}$ is linearly dependent if there exist numbers $x_1, x_2, \dots, x_k$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \dots + x_k v_k = 0.$$

This is called a *linear dependence relation* or *equation of linear dependence*.

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

has only the trivial solution, if and only if the matrix equation $Ax = 0$ has only the trivial solution, where A is the matrix with columns $v_1, v_2, \dots, v_k$:

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & \cdots & | \end{pmatrix}.$$

This is true if and only if A has a pivot position in every column.

Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \dots, v_k$ are linearly independent, or will produce a linear dependence relation by substituting any nonzero values for the free variables.

(Recall that $Ax = 0$ has a nontrivial solution if and only if A has a column without a pivot: see this observation in Section 2.4.)

Suppose that A has more columns than rows. Then A cannot have a pivot in every column (it has at most one pivot per row), so its columns are automatically linearly dependent.

A wide matrix (a matrix with more columns than rows) has linearly dependent columns.

For example, four vectors in $\mathbf{R}^3$ are automatically linearly dependent. Note that a tall matrix may or may not have linearly independent columns.

Facts about linear independence.

1. *Two vectors are linearly dependent if and only if they are collinear, i.e., one is a scalar multiple of the other.*
2. *Any set containing the zero vector is linearly dependent.*
3. *If a subset of $\{v_1, v_2, \dots, v_k\}$ is linearly dependent, then $\{v_1, v_2, \dots, v_k\}$ is linearly dependent as well.*

Proof. ^

1. If $v_1 = cv_2$ then $v_1 - cv_2 = 0$, so $\{v_1, v_2\}$ is linearly dependent. In the other direction, if $x_1 v_1 + x_2 v_2 = 0$ with $x_1 \neq 0$ (say), then $v_1 = -\frac{x_2}{x_1} v_2$.
2. It is easy to produce a linear dependence relation if one vector is the zero vector: for instance, if $v_1 = 0$ then

$$1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_k = 0.$$

3. After reordering, we may suppose that $\{v_1, v_2, \dots, v_r\}$ is linearly dependent, with $r < p$. This means that there is an equation of linear dependence

$$x_1 v_1 + x_2 v_2 + \cdots + x_r v_r = 0,$$

with at least one of $x_1, x_2, \dots, x_r$ nonzero. This is also an equation of linear dependence among $\{v_1, v_2, \dots, v_k\}$, since we can take the coefficients of $v_{r+1}, \dots, v_k$ to all be zero.

Theorem. *A set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly dependent if and only if one of the vectors is in the span of the other ones.*

Any such vector may be removed without affecting the span.

Suppose, for instance, that v_3 is in $\text{Span}\{v_1, v_2, v_4\}$, so we have an equation like

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4.$$

We can subtract v_3 from both sides of the equation to get

$$0 = 2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4.$$

This is a linear dependence relation.

In this case, any linear combination of v_1, v_2, v_3, v_4 is already a linear combination of v_1, v_2, v_4 :

$$\begin{aligned} x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 &= x_1v_1 + x_2v_2 + x_3\left(2v_1 - \frac{1}{2}v_2 + 6v_4\right) + x_4v_4 \\ &= (x_1 + 2x_3)v_1 + \left(x_2 - \frac{1}{2}x_3\right)v_2 + (x_4 + 6x_3)v_4. \end{aligned}$$

Therefore, $\text{Span}\{v_1, v_2, v_3, v_4\}$ is contained in $\text{Span}\{v_1, v_2, v_4\}$. Any linear combination of v_1, v_2, v_4 is also a linear combination of v_1, v_2, v_3, v_4 (with the v_3 -coefficient equal to zero), so $\text{Span}\{v_1, v_2, v_4\}$ is also contained in $\text{Span}\{v_1, v_2, v_3, v_4\}$, and thus they are equal.

In the other direction, if we have a linear dependence relation like

$$0 = 2v_1 - \frac{1}{2}v_2 + v_3 - 6v_4,$$

then we can move any nonzero term to the left side of the equation and divide by its coefficient:

$$v_1 = \frac{1}{2} \left(\frac{1}{2}v_2 - v_3 + 6v_4 \right).$$

This shows that v_1 is in $\text{Span}\{v_2, v_3, v_4\}$.

We leave it to the reader to generalize this proof for any set of vectors.