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Math Seminar

Reference

These notes follow the textbook *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff, School of Mathematics, Georgia Institute of Technology. Definitions, examples and figures in these slides are taken or adapted from it.

<https://textbooks.math.gatech.edu/ila/>

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Subspaces: Definition and Examples

Definition. A *subset* of $\mathbf{R}^n$ is any collection of points of $\mathbf{R}^n$.

For instance, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$.

Above we expressed C in set builder notation: in English, it reads “ C is the set of all ordered pairs (x, y) in $\mathbf{R}^2$ such that $x^2 + y^2 = 1$.”

Definition. A *subspace* of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. ***Non-emptiness:*** The zero vector is in V .
2. ***Closure under addition:*** If u and v are in V , then $u + v$ is also in V .
3. ***Closure under scalar multiplication:*** If v is in V and c is in $\mathbf{R}$, then cv is also in V .

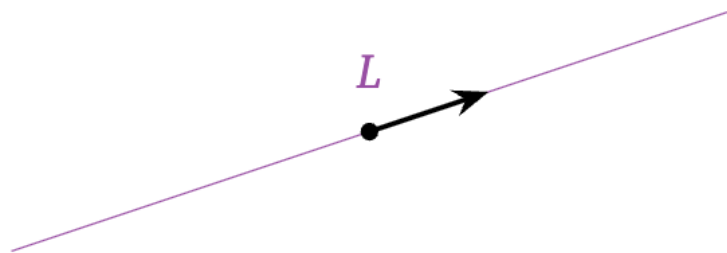
As a consequence of these properties, we see:

- If v is a vector in V , then all scalar multiples of v are in V by the third property. In other words the line through any nonzero vector in V is also contained in V .
- If u, v are vectors in V and c, d are scalars, then cu, dv are also in V by the third property, so $cu + dv$ is in V by the second property. Therefore, all of $\text{Span}\{u, v\}$ is contained in V .
- Similarly, if $v_1, v_2, \dots, v_n$ are all in V , then $\text{Span}\{v_1, v_2, \dots, v_n\}$ is contained in V . In other words, *a subspace contains the span of any vectors in it.*

Example. The set $\mathbf{R}^n$ is a subspace of itself: indeed, it contains zero, and is closed under addition and scalar multiplication.

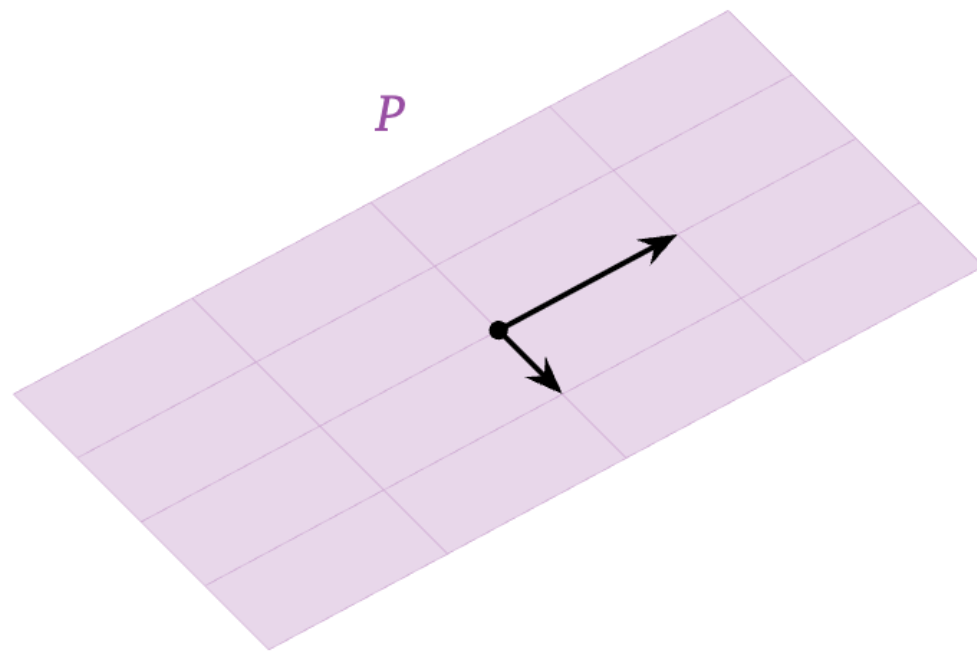
Example. The set $\{0\}$ containing only the zero vector is a subspace of $\mathbf{R}^n$: it contains zero, and if you add zero to itself or multiply it by a scalar, you always get zero.

A line L through the origin is a subspace.



Indeed, L contains zero, and is easily seen to be closed under addition and scalar multiplication.

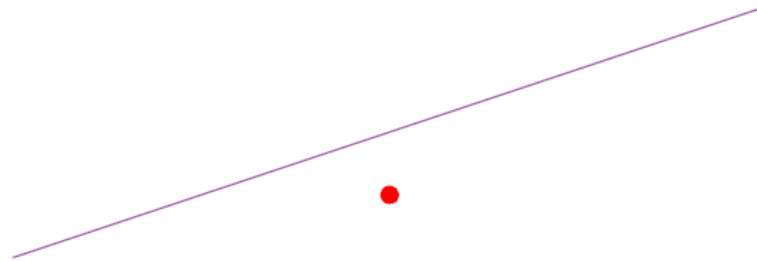
A plane P through the origin is a subspace.



Indeed, P contains zero; the sum of two vectors in P is also in P ; and any scalar multiple of a vector in P is also in P .

Non-example (A line not containing the origin). ^

A line L (or any other subset) that does not contain the origin is not a subspace. It fails the first defining property: *every subspace contains the origin* by definition.



Theorem (Spans are Subspaces and Subspaces are Spans). *If $v_1, v_2, \dots, v_p$ are any vectors in $\mathbf{R}^n$, then $\text{Span}\{v_1, v_2, \dots, v_p\}$ is a subspace of $\mathbf{R}^n$. Moreover, any subspace of $\mathbf{R}^n$ can be written as a span of a set of p linearly independent vectors in $\mathbf{R}^n$ for $p \leq n$.*

Proof: To show that $\text{Span}\{v_1, v_2, \dots, v_p\}$ is a subspace, we have to verify the three defining properties.

1. The zero vector $0 = 0v_1 + 0v_2 + \dots + 0v_p$ is in the span.
2. If $u = a_1v_1 + a_2v_2 + \dots + a_pv_p$ and $v = b_1v_1 + b_2v_2 + \dots + b_pv_p$ are in $\text{Span}\{v_1, v_2, \dots, v_p\}$, then

$$u + v = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \dots + (a_p + b_p)v_p$$

is also in $\text{Span}\{v_1, v_2, \dots, v_p\}$.

3. If $v = a_1v_1 + a_2v_2 + \dots + a_pv_p$ is in $\text{Span}\{v_1, v_2, \dots, v_p\}$ and c is a scalar, then

$$cv = ca_1v_1 + ca_2v_2 + \dots + ca_pv_p$$

is also in $\text{Span}\{v_1, v_2, \dots, v_p\}$.

Since $\text{Span}\{v_1, v_2, \dots, v_p\}$ satisfies the three defining properties of a subspace, it is a subspace.

Proof(continued):

Now let V be a subspace of $\mathbf{R}^n$. If V is the zero subspace, then it is the span of the empty set, so we may assume V is nonzero. Choose a nonzero vector v_1 in V . If $V = \text{Span}\{v_1\}$, then we are done. Otherwise, there exists a vector v_2 that is in V but not in $\text{Span}\{v_1\}$. Then $\text{Span}\{v_1, v_2\}$ is contained in V , and by the increasing span criterion in Section 2.5, the set $\{v_1, v_2\}$ is linearly independent. If $V = \text{Span}\{v_1, v_2\}$ then we are done. Otherwise, we continue in this fashion until we have written $V = \text{Span}\{v_1, v_2, \dots, v_p\}$ for some linearly independent set $\{v_1, v_2, \dots, v_p\}$. This process terminates after at most n steps by this important note in Section 2.5.

If $V = \text{Span}\{v_1, v_2, \dots, v_p\}$, we say that V is the subspace ***spanned by*** or ***generated by*** the vectors $v_1, v_2, \dots, v_p$. We call $\{v_1, v_2, \dots, v_p\}$ a ***spanning set*** for V .

Definition. Let A be an $m \times n$ matrix.

- The ***column space*** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A . It is written $\text{Col}(A)$.
- The ***null space*** of A is the subspace of $\mathbf{R}^n$ consisting of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul}(A) = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

Proof that the null space of a matrix A is a subspace.

We have to verify the three defining properties. To say that a vector v is in $\text{Nul}(A)$ means that $Av = 0$.

1. The zero vector is in $\text{Nul}(A)$ because $A0 = 0$.
2. Suppose that u, v are in $\text{Nul}(A)$. This means that $Au = 0$ and $Av = 0$. Hence

$$A(u + v) = Au + Av = 0 + 0 = 0$$

by the linearity of the matrix-vector product in Section 2.3. Therefore, $u + v$ is in $\text{Nul}(A)$.

3. Suppose that v is in $\text{Nul}(A)$ and c is a scalar. Then

$$A(cv) = cAv = c \cdot 0 = 0$$

by the linearity of the matrix-vector product in Section 2.3, so cv is also in $\text{Nul}(A)$.

Since $\text{Nul}(A)$ satisfies the three defining properties of a subspace, it is a subspace.

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Recipe: Compute a spanning set for a null space.

To find a spanning set for $\text{Nul}(A)$, compute the parametric vector form of the solutions to the homogeneous equation $Ax = 0$. The vectors attached to the free variables form a spanning set for $\text{Nul}(A)$.

Find a spanning set for the null space of the matrix

$$A = \begin{pmatrix} 2 & 3 & -8 & -5 \\ -1 & 2 & -3 & -8 \end{pmatrix}.$$

Solution

We compute the parametric vector form of the solutions of $Ax = 0$. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & -3 \end{pmatrix}.$$

The free variables are x_3 and x_4 ; the parametric form of the solution set is

$$\begin{cases} x_1 = x_3 - 2x_4 \\ x_2 = 2x_3 + 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \xrightarrow[\text{vector form}]{\text{parametric}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore,

$$\text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

 **Definition.** Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \dots, v_m\}$, and
2. the set $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

Recall that a set of vectors is *linearly independent* if and only if, when you remove any vector from the set, the span shrinks (Theorem 2.5.12). In other words, if $\{v_1, v_2, \dots, v_m\}$ is a basis of a subspace V , then no proper subset of $\{v_1, v_2, \dots, v_m\}$ will span V : it is a *minimal* spanning set. Any subspace admits a basis by this theorem in Section 2.6.

A nonzero subspace has *infinitely many* different bases, but they all contain the same number of vectors.

 **Definition.** Let V be a subspace of $\mathbf{R}^n$. The number of vectors in any basis of V is called the ***dimension*** of V , and is written $\dim V$.

Example. The previous example implies that any basis for $\mathbf{R}^n$ has n vectors in it. Let $v_1, v_2, \dots, v_n$ be vectors in $\mathbf{R}^n$, and let A be the $n \times n$ matrix with columns $v_1, v_2, \dots, v_n$.

1. To say that $\{v_1, v_2, \dots, v_n\}$ spans $\mathbf{R}^n$ means that A has a pivot position in every *row*: see this theorem in Section 2.3.
2. To say that $\{v_1, v_2, \dots, v_n\}$ is linearly independent means that A has a pivot position in every *column*: see this important note in Section 2.5.

A basis for the column space. First we show how to compute a basis for the column space of a matrix.

Theorem. *The pivot columns of a matrix A form a basis for $\text{Col}(A)$.*

The above theorem is referring to the pivot columns in the *original* matrix, not its reduced row echelon form. Indeed, a matrix and its reduced row echelon form generally have different column spaces. For example, in the matrix A below:

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis
pivot columns in RREF

the pivot columns are the first two columns, so a basis for $\text{Col}(A)$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Corollary. *The dimension of $\text{Col}(A)$ is the number of pivots of A .*

A basis for the null space. In order to compute a basis for the null space of a matrix, one has to find the parametric vector form of the solutions of the homogeneous equation $Ax = 0$.

Theorem. *The vectors attached to the free variables in the parametric vector form of the solution set of $Ax = 0$ form a basis of $\text{Nul}(A)$.*

The Basis Theorem

Recall that $\{v_1, v_2, \dots, v_n\}$ forms a basis for $\mathbf{R}^n$ if and only if the matrix A with columns $v_1, v_2, \dots, v_n$ has a pivot in every row and column (see this [example](#)). Since A is an $n \times n$ matrix, these two conditions are equivalent: the vectors span if and only if they are linearly independent. The basis theorem is an abstract version of the preceding statement, that applies to any subspace.

Basis Theorem. *Let V be a subspace of dimension m . Then:*

- *Any m linearly independent vectors in V form a basis for V .*
- *Any m vectors that span V form a basis for V .*

Find a basis for the plane

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 + x_2 = x_3 \right\}$$

by inspection. (This plane is expressed in set builder notation.)

Solution

First note that V is the null space of the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$; this matrix is in reduced row echelon form and has two free variables, so V is indeed a plane. We write down two vectors satisfying $x_1 + x_2 = x_3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Since v_1 and v_2 are not collinear, they are linearly independent; since $\dim(V) = 2$, the basis theorem implies that $\{v_1, v_2\}$ is a basis for V .

Definition. The *rank* of a matrix A , written $\text{rank}(A)$, is the dimension of the column space $\text{Col}(A)$.

The *nullity* of a matrix A , written $\text{nullity}(A)$, is the dimension of the null space $\text{Nul}(A)$.

$$\begin{aligned}\text{rank}(A) &= \dim \text{Col}(A) = \text{the number of columns with pivots} \\ \text{nullity}(A) &= \dim \text{Nul}(A) = \text{the number of free variables} \\ &= \text{the number of columns without pivots}\end{aligned}$$

Clearly

$$\#(\text{columns with pivots}) + \#(\text{columns without pivots}) = \#(\text{columns}),$$

Rank Theorem. *If A is a matrix with n columns, then*

$$\text{rank}(A) + \text{nullity}(A) = n.$$

In other words, for any consistent system of linear equations,

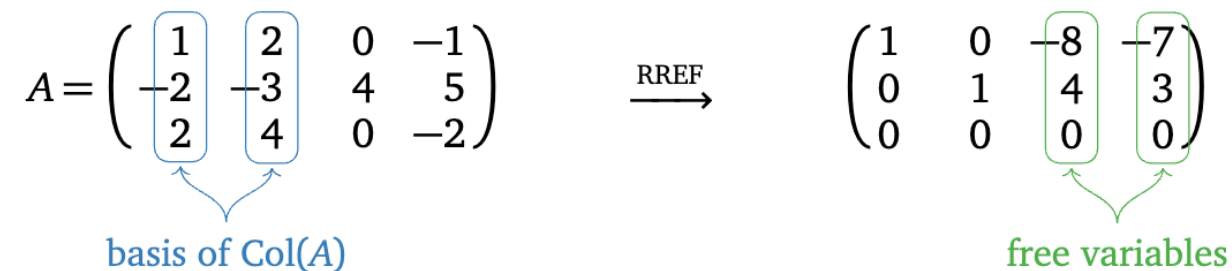
$$(\text{dim of column span}) + (\text{dim of solution set}) = (\text{number of variables}).$$

$$Ax = 0$$

The rank theorem theorem is really the culmination of this chapter, as it gives a strong relationship between the null space of a matrix (the solution set of $Ax = 0$) with the column space (the set of vectors b making $Ax = b$ consistent), our two primary objects of interest. The more freedom we have in choosing x the less freedom we have in choosing b and vice versa.

Consider the following matrix and its reduced row echelon form:

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 0 & -1 \\ -2 & -3 & 4 & 5 \\ \boxed{2} & \boxed{4} & 0 & -2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}$$



basis of $\text{Col}(A)$ free variables

A basis for $\text{Col}(A)$ is given by the pivot columns:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\},$$

so $\text{rank}(A) = \dim \text{Col}(A) = 2$.

Since there are two free variables x_3, x_4 , the null space of A has two vectors (see this [theorem in Section 2.7](#)):

$$\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\},$$

so $\text{nullity}(A) = 2$.

In this case, the rank theorem says that $2 + 2 = 4$, where 4 is the number of columns.