

Linear Algebra Notes

Lecture 5:

Coordinates

Math Seminar

Reference

These notes follow the textbook *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff, School of Mathematics, Georgia Institute of Technology. Definitions, examples and figures in these slides are taken or adapted from it.

https://textbooks.math.gatech.edu/ila/

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Fact. *If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V, then any vector x in V can be written as a linear combination*

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in exactly one way.

Proof. Recall that to say $\mathcal{B}$ is a *basis* for V means that $\mathcal{B}$ spans V and $\mathcal{B}$ is linearly independent. Since $\mathcal{B}$ spans V, we can write any x in V as a linear combination of $v_1, v_2, \ldots, v_m$. For uniqueness, suppose that we had two such expressions:

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$
$$x = c_1' v_1 + c_2' v_2 + \cdots + c_m' v_m.$$

Subtracting the first equation from the second yields

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m.$$

Since $\mathcal{B}$ is linearly independent, the only solution to the above equation is the trivial solution: all the coefficients must be zero. It follows that $c_i - c_i'$ for all i, which proves that $c_1 = c_1'$, $c_2 = c_2'$, $\ldots$, $c_m = c_m'$.

Example. Consider the standard basis of $\mathbf{R}^3$ from this example in Section 2.7:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

According to the above fact, every vector in $\mathbf{R}^3$ can be written as a linear combination of e_1, e_2, e_3, with unique coefficients. For example,

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

In this case, *the coordinates of v are exactly the coefficients of e_1, e_2, e_3.*

Definition. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace V, and let

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

be a vector in V. The coefficients $c_1, c_2, \ldots, c_m$ are the ***coordinates of x with respect to $\mathcal{B}$***. The ***$\mathcal{B}$-coordinate vector of x*** is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

If we change the basis, then we can still give instructions for how to get to the point $(3, 5, -2)$, but the instructions will be different. Say for example we take the basis

$$v_1 = e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can write $(3, 5, -2)$ in this basis as $3v_1 + 2v_2 - 2v_3$. In other words: start at the origin, travel northeast 3 times as far as v_1, then 2 units east, then 2 units down. In this situation, we can say that "3 is the v_1-coordinate of $(3, 5, -2)$, 2 is the v_2-coordinate of $(3, 5, -2)$, and -2 is the v_3-coordinate of $(3, 5, -2)$."

Define

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathcal{B} = \{v_1, v_2\}, \quad V = \text{Span}\{v_1, v_2\}.$$

1. Verify that $\mathcal{B}$ is a basis for V.
2. If $[w]_{\mathcal{B}} = \binom{5}{2}$, then what is w?
3. Find the $\mathcal{B}$-coordinates of $v = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

Solution

1. We need to verify that $\mathcal{B}$ spans V, and that it is linearly independent. By definition, V is the span of $\mathcal{B}$; since v_1 and v_2 are not multiples of each other, they are linearly independent. This shows in particular that V is a *plane*.

2. To say $[w]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ means that 5 is the v_1-coordinate of w, and that 2 is the v_2-coordinate:

$$w = 5v_1 + 2v_2 = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix}.$$

3. We have to solve the vector equation $v = c_1v_1 + c_2v_2$ in the unknowns c_1, c_2. We form an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right).$$

We have $c_1 = 2$ and $c_2 = 3$, so $v = 2v_1 + 3v_2$ and $[v]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Define

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix},\ v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad V = \text{Span}\{v_1, v_2, v_3\}.$$

1. Find a basis $\mathcal{B}$ for V.
2. Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

Solution

1. We write V as the column space of a matrix A, then row reduce to find the pivot columns, as in this example in Section 2.7.

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first two columns are pivot columns, so we can take $\mathcal{B} = \{v_1, v_2\}$ as our basis for V.

2. We have to solve the vector equation $x = c_1 v_1 + c_2 v_2$. We form an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

We have $c_1 = 3$ and $c_2 = 2$, so $x = 3v_1 + 2v_2$, and thus $[x]_{\mathcal{B}} = \binom{3}{2}$.

Recipes: $\mathcal{B}$-coordinates.

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

Finding the $\mathcal{B}$-coordinates of x means solving the vector equation

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in the unknowns $c_1, c_2, \ldots, c_m$. This generally means row reducing the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array}\right).$$