

# Linear Algebra Notes

## Lecture 6:

### Transformations

### Math Seminar

# Reference

These notes follow the textbook *Interactive Linear Algebra* by Dan Margalit and Joseph Rabinoff, School of Mathematics, Georgia Institute of Technology. Definitions, examples and figures in these slides are taken or adapted from it.

https://textbooks.math.gatech.edu/ila/

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# Transformations ¶ permalink

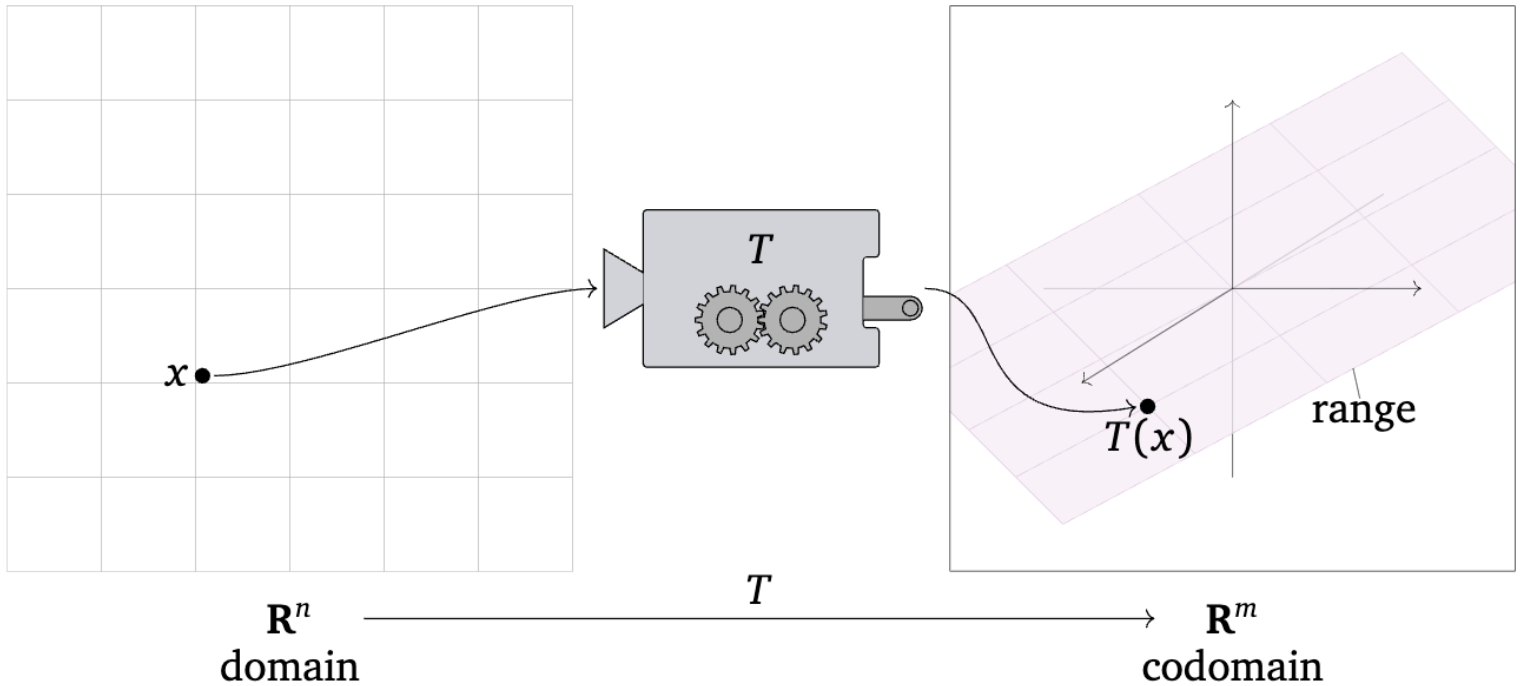
At this point it is convenient to fix our ideas and terminology regarding functions, which we will call *transformations* in this book. This allows us to systematize our discussion of matrices as functions.

**Definition.** A ***transformation*** from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule $T$ that assigns to each vector $x$ in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the ***domain*** of $T$.
- $\mathbf{R}^m$ is called the ***codomain*** of $T$.
- For $x$ in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the ***image*** of $x$ under $T$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the ***range*** of $T$.

The notation $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means "$T$ is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$."

It may help to think of $T$ as a "machine" that takes $x$ as an input, and gives you $T(x)$ as the output.




The points of the domain $\mathbf{R}^n$ are the *inputs* of $T$: this simply means that it makes sense to evaluate $T$ on vectors with $n$ entries, i.e., lists of $n$ numbers. Likewise, the points of the codomain $\mathbf{R}^m$ are the *outputs* of $T$: this means that the result of evaluating $T$ is always a vector with $m$ entries.

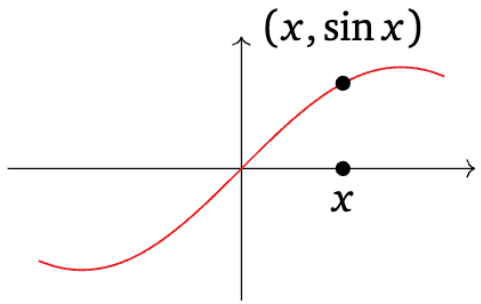
The *range* of $T$ is the set of all vectors in the codomain that actually arise as outputs of the function $T$, for some input. In other words, the range is all vectors $b$ in the codomain such that $T(x) = b$ has a solution $x$ in the domain.

Most of the functions you may have seen previously have domain and codomain equal to $\mathbf{R} = \mathbf{R}^1$. For example,

$$\sin\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad \sin(x) = \begin{pmatrix} \text{the length of the opposite} \\ \text{edge over the hypotenuse of} \\ \text{a right triangle with angle } x \\ \text{in radians} \end{pmatrix}.$$

Notice that we have defined sin by a rule: a function is defined by specifying what the output of the function is for any possible input.

You may be used to thinking of such functions in terms of their graphs:




In this case, the horizontal axis is the domain, and the vertical axis is the codomain. This is useful when the domain and codomain are $\mathbf{R}$, but it is hard to do when, for instance, the domain is $\mathbf{R}^2$ and the codomain is $\mathbf{R}^3$. The graph of such a function is a subset of $\mathbf{R}^5$, which is difficult to visualize. For this reason, we will rarely graph a transformation.

Note that the *range* of sin is the interval $[-1, 1]$: this is the set of all possible outputs of the sin function.

**Definition.** Let $A$ be an $m \times n$ matrix. The ***matrix transformation*** associated to $A$ is the transformation

$$T : \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

This is the transformation that takes a vector $x$ in $\mathbf{R}^n$ to the vector $Ax$ in $\mathbf{R}^m$.

If $A$ has $n$ columns, then it only makes sense to multiply $A$ by vectors with $n$ entries. This is why the domain of $T(x) = Ax$ is $\mathbf{R}^n$. If $A$ has $n$ rows, then $Ax$ has $m$ entries for any vector $x$ in $\mathbf{R}^n$; this is why the codomain of $T(x) = Ax$ is $\mathbf{R}^m$.

The definition of a matrix transformation $T$ tells us how to evaluate $T$ on any given vector: we multiply the input vector by a matrix. For instance, let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

and let $T(x) = Ax$ be the associated matrix transformation. Then

$$T\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = A\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \end{pmatrix}.$$

Suppose that $A$ has columns $v_1, v_2, \ldots, v_n$. If we multiply $A$ by a general vector $x$, we get

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

This is just a general linear combination of $v_1, v_2, \ldots, v_n$. Therefore, the outputs of $T(x) = Ax$ are exactly the linear combinations of the columns of $A$: the *range* of $T$ is the column space of $A$. See this note in Section 2.3.

Let $A$ be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation.

- The *domain* of $T$ is $\mathbf{R}^n$, where $n$ is the number of *columns* of $A$.
- The *codomain* of $T$ is $\mathbf{R}^m$, where $m$ is the number of *rows* of $A$.
- The *range* of $T$ is the *column space* of $A$.

Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & 4 \end{pmatrix},$$

and define $T(x) = Ax$. The domain of $T$ is $\mathbf{R}^3$, and the codomain is $\mathbf{R}^2$. The range of $T$ is the column space; since all three columns are collinear, the range is a line in $\mathbf{R}^2$.

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define $T(x) = Ax$. The domain of $T$ is $\mathbf{R}^2$, and the codomain is $\mathbf{R}^3$. The range of $T$ is the column space; since $A$ has two columns which are not collinear, the range is a plane in $\mathbf{R}^3$.

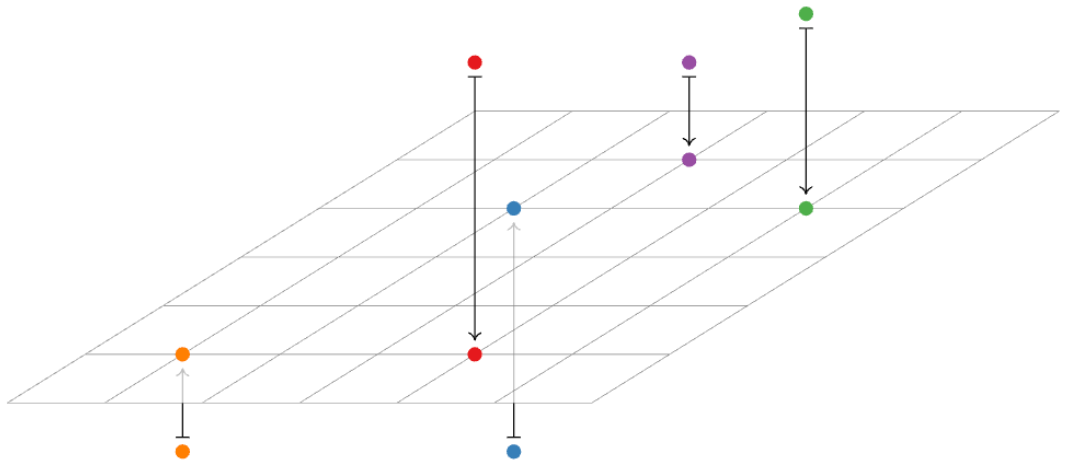
Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and let $T(x) = Ax$. What are the domain, the codomain, and the range of $T$?

**Solution**

Geometrically, the transformation $T$ projects a vector directly "down" onto the $xy$-plane in $\mathbf{R}^3$.



The inputs and outputs have three entries, so the domain and codomain are both $\mathbf{R}^3$. The possible outputs all lie on the $xy$-plane, and every point on the $xy$-plane is an output of $T$ (with itself as the input), so the range of $T$ is the $xy$-plane.

Be careful not to confuse the codomain with the range here. The range is a plane, but it is a plane *in* $\mathbf{R}^3$, so the codomain is still $\mathbf{R}^3$. The outputs of $T$ all have three entries; the last entry is simply always zero.

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a matrix transformation.

1. Evaluate $T(u)$ for $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
2. Let

$$b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

   Find a vector $v$ in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?
3. Does there exist a vector $w$ in $\mathbf{R}^3$ such that there is more than one $v$ in $\mathbf{R}^2$ with $T(v) = w$?
4. Find a vector $w$ in $\mathbf{R}^3$ which is not in the range of $T$.

**Solution**

1. We evaluate $T(u)$ by substituting the definition of $T$ in terms of matrix multiplication:

$$T\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}1&1\\0&1\\1&1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}7\\4\\7\end{pmatrix}.$$

2. We want to find a vector $v$ such that $b=T(v)=Av$. In other words, we want to solve the matrix equation $Av=b$. We form an augmented matrix and row reduce:

$$\begin{pmatrix}1&1\\0&1\\1&1\end{pmatrix}v=\begin{pmatrix}7\\5\\7\end{pmatrix}\xrightarrow[\text{matrix}]{\text{augmented}}\left(\begin{array}{cc|c}1&1&7\\0&1&5\\1&1&7\end{array}\right)\xrightarrow{\text{row reduce}}\left(\begin{array}{cc|c}1&0&2\\0&1&5\\0&0&0\end{array}\right).$$

This gives $x=2$ and $y=5$, so that there is a unique vector

$$v=\begin{pmatrix}2\\5\end{pmatrix}$$

such that $T(v)=b$.

3. *Translation:* is there any vector $w$ in $\mathbf{R}^3$ such that the solution set of $Av = w$ has more than one vector in it? The solution set of $Ax = w$, if non empty, is a translate of the solution set of $Av = b$ above, which has one vector in it. See this key observation in Section 2.4. It follows that the solution set of $Av = w$ can have at most one vector.

4. *Translation:* find a vector $w$ such that the matrix equation $Av = w$ is not consistent. Notice that if we take

$$w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

then the matrix equation $Av = w$ translates into the system of equations

$$\begin{cases} x + y = 1 \\ \quad y = 2 \\ x + y = 3, \end{cases}$$

which is clearly inconsistent.

Define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by the formula

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \ln(x) \\ \cos(y) \\ \ln(x) \end{pmatrix}.$$

1. Evaluate $T(u)$ for $u = \begin{pmatrix} 1 \\ \pi \end{pmatrix}$.
2. Let

$$b = \begin{pmatrix} 7 \\ 1 \\ 7 \end{pmatrix}.$$

   Find a vector $v$ in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?
3. Does there exist a vector $w$ in $\mathbf{R}^3$ such that there is more than one $v$ in $\mathbf{R}^2$ with $T(v) = w$?
4. Find a vector $w$ in $\mathbf{R}^3$ which is not in the range of $T$.

**Solution**

1. We evaluate $T(u)$ using the defining formula:

$$T\begin{pmatrix}1\\ \pi\end{pmatrix} = \begin{pmatrix}\ln(1)\\ \cos(\pi)\\ \ln(1)\end{pmatrix} = \begin{pmatrix}0\\ -1\\ 0\end{pmatrix}.$$

2. We have

$$T\begin{pmatrix}e^7\\ 2\pi n\\ e^7\end{pmatrix} = \begin{pmatrix}\ln(e^7)\\ \cos(2\pi n)\\ \ln(e^7)\end{pmatrix} = \begin{pmatrix}7\\ 1\\ 7\end{pmatrix}$$

   for any whole number $n$. Hence there are infinitely many such vectors.

3. The vector $b$ from the previous part is an example of such a vector.

4. Since $\cos(y)$ is always between $-1$ and $1$, the vector

$$w = \begin{pmatrix}0\\ 2\\ 0\end{pmatrix}$$

   is not in the range of $T$.

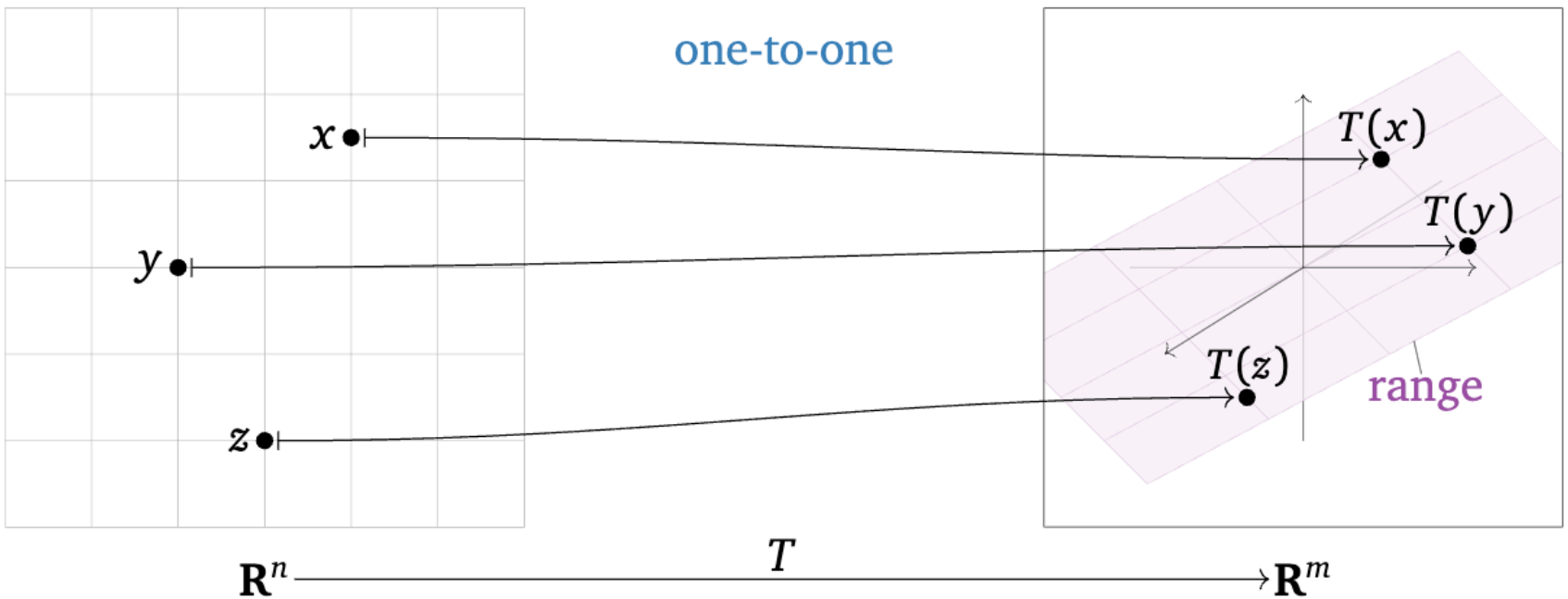
# One-to-one Transformations

**Definition (One-to-one transformations).** A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is ***one-to-one*** if, for every vector $b$ in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution $x$ in $\mathbf{R}^n$.

Another word for *one-to-one* is *injective*.

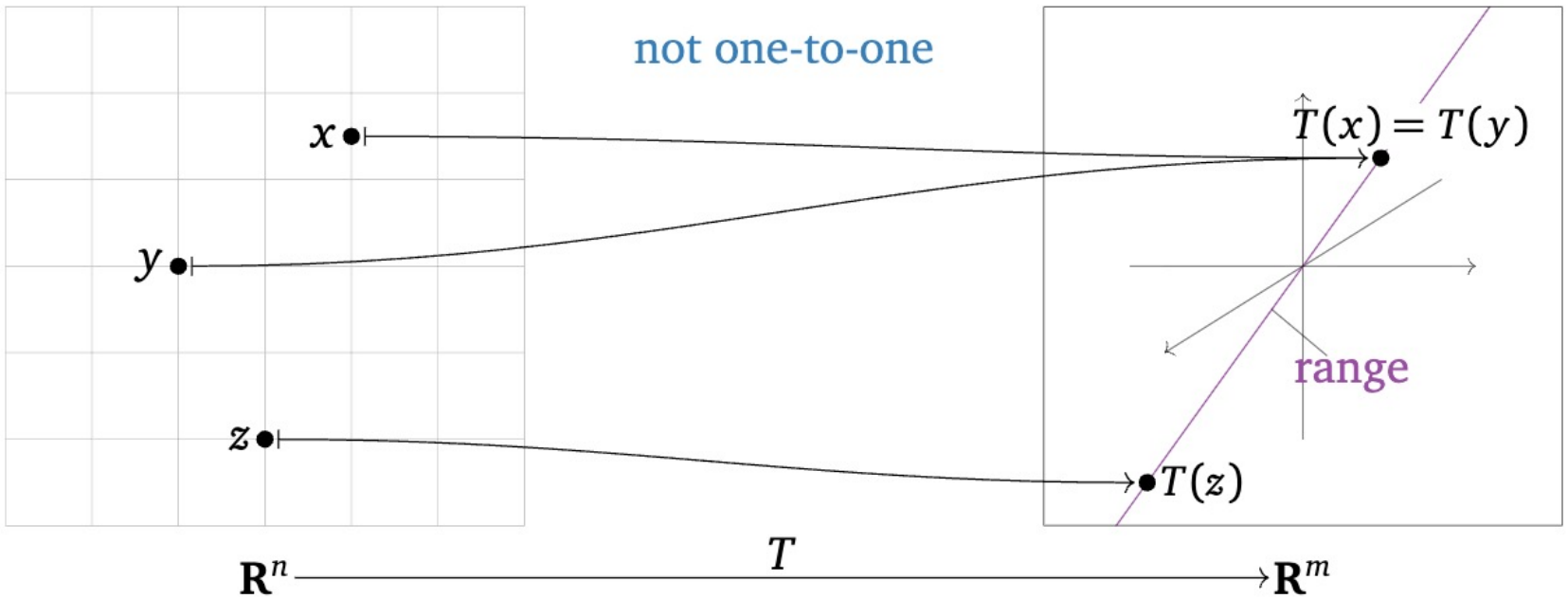
Here are some equivalent ways of saying that $T$ is one-to-one:

- For every vector $b$ in $\mathbf{R}^m$, the equation $T(x) = b$ has *zero or one* solution $x$ in $\mathbf{R}^n$.
- Different inputs of $T$ have different outputs.
- If $T(u) = T(v)$ then $u = v$.



Here are some equivalent ways of saying that $T$ is *not* one-to-one:

- There exists some vector $b$ in $\mathbf{R}^m$ such that the equation $T(x) = b$ has *more than one* solution $x$ in $\mathbf{R}^n$.
- There are two different inputs of $T$ with the same output.
- There exist vectors $u, v$ such that $u \neq v$ but $T(u) = T(v)$.



The function $\sin\colon \mathbf{R} \to \mathbf{R}$ is not one-to-one. Indeed, $\sin(0) = \sin(\pi) = 0$, so the inputs 0 and $\pi$ have the same output 0. In fact, the equation $\sin(x) = 0$ has infinitely many solutions $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$.

The function $\exp\colon \mathbf{R} \to \mathbf{R}$ defined by $\exp(x) = e^x$ is one-to-one. Indeed, if $T(x) = T(y)$, then $e^x = e^y$, so $\ln(e^x) = \ln(e^y)$, and hence $x = y$. The equation $T(x) = C$ has one solution $x = \ln(C)$ if $C > 0$, and it has zero solutions if $C \leq 0$.

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$ is one-to-one. Indeed, if $f(x) = f(y)$ then $x^3 = y^3$; taking cube roots gives $x = y$. In other words, the only solution of $f(x) = C$ is $x = \sqrt[3]{C}$.

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3 - x$ is not one-to-one. Indeed, $f(0) = f(1) = f(-1) = 0$, so the inputs $0, 1, -1$ all have the same output 0. The solutions of the equation $x^3 - x = 0$ are exactly the roots of $f(x) = x(x-1)(x+1)$, and this equation has three roots.

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ is not one-to-one. Indeed, $f(1) = 1 = f(-1)$, so the inputs 1 and $-1$ have the same outputs. The function $g\colon \mathbf{R} \to \mathbf{R}$ defined by $g(x) = |x|$ is not one-to-one for the same reason.

**Theorem (One-to-one matrix transformations).** *Let $A$ be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:*

1. *$T$ is one-to-one.*
2. *For every $b$ in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.*
3. *For every $b$ in $\mathbf{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.*
4. *$Ax = 0$ has only the trivial solution.*
5. *The columns of $A$ are linearly independent.*
6. *$A$ has a pivot in every column.*
7. *The range of $T$ has dimension $n$.*

Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(x) = Ax$. Is $T$ one-to-one?

**Solution**

The reduced row echelon form of $A$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence $A$ has a pivot in every column, so $T$ is one-to-one.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T(x) = Ax$. Is $T$ one-to-one? If not, find two different vectors $u, v$ such that $T(u) = T(v)$.

**Solution**

The matrix $A$ is already in reduced row echelon form. It does not have a pivot in every column, so $T$ is not one-to-one. Therefore, we know from the theorem that $Ax = 0$ has nontrivial solutions. If $v$ is a nontrivial (i.e., nonzero) solution of $Av = 0$, then $T(v) = Av = 0 = A0 = T(0)$, so 0 and $v$ are different vectors with the same output. For instance,

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 = T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Geometrically, $T$ is projection onto the $xy$-plane. Any two vectors that lie on the same vertical line will have the same projection. For $b$ on the $xy$-plane, the solution set of $T(x) = b$ is the entire vertical line containing $b$. In particular, $T(x) = b$ has *infinitely many* solutions.

**Wide matrices do not have one-to-one transformations.** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one matrix transformation, what can we say about the relative sizes of $n$ and $m$?

The matrix associated to $T$ has $n$ columns and $m$ rows. Each row and each column can only contain one pivot, so in order for $A$ to have a pivot in every column, it must have *at least as many rows* as columns: $n \leq m$.

This says that, for instance, $\mathbf{R}^3$ is "too big" to admit a one-to-one linear transformation into $\mathbf{R}^2$.

Note that there exist tall matrices that are not one-to-one: for example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

does not have a pivot in every column.

# Onto Transformations

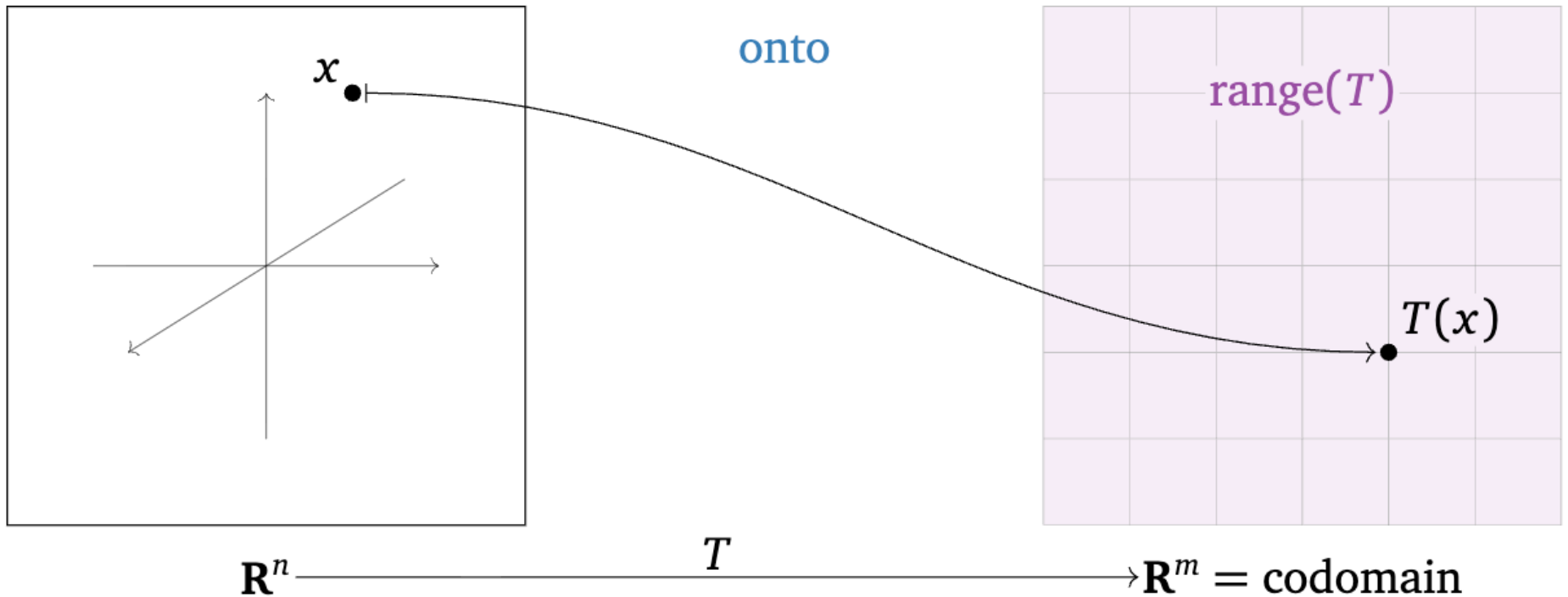
**Definition (Onto transformations).** A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is ***onto*** if, for every vector $b$ in $\mathbf{R}^m$, the equation $T(x) = b$ has *at least one* solution $x$ in $\mathbf{R}^n$.

**Remark.** ^

Another word for *onto* is *surjective*.

Here are some equivalent ways of saying that $T$ is onto:

- The range of $T$ is equal to the codomain of $T$.
- Every vector in the codomain is the output of some input vector.



**Theorem (Onto matrix transformations).** *Let $A$ be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. The following statements are equivalent:*

1. *$T$ is onto.*
2. *$T(x) = b$ has at least one solution for every $b$ in $\mathbf{R}^m$.*
3. *$Ax = b$ is consistent for every $b$ in $\mathbf{R}^m$.*
4. *The columns of $A$ span $\mathbf{R}^m$.*
5. *$A$ has a pivot in every row.*
6. *The range of $T$ has dimension $m$.*

Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x) = Ax$. Is $T$ onto?

**Solution**

The reduced row echelon form of $A$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence $A$ has a pivot in every row, so $T$ is onto.

Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(x) = Ax$. Is $T$ onto? If not, find a vector $b$ in $\mathbf{R}^3$ such that $T(x) = b$ has no solution.

**Solution**

The reduced row echelon form of $A$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence $A$ does not have a pivot in every row, so $T$ is not onto. In fact, since

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x \end{pmatrix},$$

we see that for every output vector of $T$, the third entry is equal to the first. Therefore,

$$b = (1, 2, 3)$$

is not in the range of $T$.

The previous two examples illustrate the following observation. Suppose that $T(x) = Ax$ is a matrix transformation that is *not* onto. This means that $\text{range}(T) = \text{Col}(A)$ is a subspace of $\mathbf{R}^m$ of dimension less than $m$: perhaps it is a line in the plane, or a line in 3-space, or a plane in 3-space, etc. Whatever the case, the range of $T$ is *very small* compared to the codomain. To find a vector not in the range of $T$, choose a random nonzero vector $b$ in $\mathbf{R}^m$; you have to be extremely unlucky to choose a vector that is in the range of $T$. Of course, to check whether a given vector $b$ is in the range of $T$, you have to solve the matrix equation $Ax = b$ to see whether it is consistent.

**Tall matrices do not have onto transformations.** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is an onto matrix transformation, what can we say about the relative sizes of $n$ and $m$?

The matrix associated to $T$ has $n$ columns and $m$ rows. Each row and each column can only contain one pivot, so in order for $A$ to have a pivot in every row, it must have *at least as many columns* as rows: $m \leq n$.

This says that, for instance, $\mathbf{R}^2$ is "too small" to admit an onto linear transformation to $\mathbf{R}^3$.

Note that there exist wide matrices that are not onto: for example,

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

does not have a pivot in every row.

# Comparison

The above expositions of one-to-one and onto transformations were written to mirror each other. However, "one-to-one" and "onto" are complementary notions: neither one implies the other. Below we have provided a chart for comparing the two. In the chart, $A$ is an $m \times n$ matrix, and $T : \mathbf{R}^n \to \mathbf{R}^m$ is the matrix transformation $T(x) = Ax$.

| $T$ **is one-to-one** | $T$ **is onto** |
| --- | --- |
| $T(x) = b$ has *at most one solution* for every $b$. | $T(x) = b$ has *at least one solution* for every $b$. |
| The columns of $A$ are linearly independent. | The columns of $A$ span $\mathbf{R}^m$. |
| $A$ has a pivot in every column. | $A$ has a pivot in every row. |
| The range of $T$ has dimension $n$. | The range of $T$ has dimension $m$. |

**One-to-one is the same as onto for square matrices.** We observed in the previous example that a square matrix has a pivot in every row if and only if it has a pivot in every column. Therefore, a matrix transformation $T$ from $\mathbf{R}^n$ to itself is one-to-one if and only if it is onto: in this case, the two notions are equivalent.

Conversely, by this note and this note, if a matrix transformation $T : \mathbf{R}^m \to \mathbf{R}^n$ is both one-to-one and onto, then $m = n$.

Note that in general, a transformation $T$ is both one-to-one and onto if and only if $T(x) = b$ has *exactly one* solution for all $b$ in $\mathbf{R}^m$.